Reverse Engineering a Protractor, Triangle Paths

UNIT

16

Mathematical Concepts

- An angle is a rotation — a turn.
- Turn angles are measured as parts of whole turns.
- A degree is $1/360^{th}$ of a whole turn.
- A protractor is a tool for measuring angles.
- A path can be composed of measured straight-line segments and measured turns.
- A path beginning and ending at the same point composed of 3 straight segments and 3 turns is called a triangle.
- The sum of the lengths of any two of the sides of a triangle must exceed the length of the third side. This is a consequence of straight.

Unit Overview

Students reverse engineer a protractor and then to use the protractor to construct triangle paths. Reverse engineering the protractor begins by considering parts of whole turns, $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{3}, \frac{1}{6}$, and by marking these parts on a circle. Relations among these parts of whole turns are established, followed by introduction of a degree as $1/360^{th}$ of a whole turn. Each part-unit of a whole turn is related to its corresponding degree measure. Students then use conventional circular protractors to create turn paths with different degree measures. Students are challenged to create a turning path on paper that begins and ends at the same point, with 3 straight sides, all of the same length, and 3 turns. Students write directions and consider how to modify them to create a larger or smaller equilateral triangle. During the next phase, students write directions to create a triangle that is not equilateral. Students use their directions to create a triangular ribbon path outside, using flags, ribbon (surveyor tape) and a yardstick. With partners, and during whole group discussion, students consider the triangle inequality, the sums of the exterior (turn) angles and the interior angles.

Contents

Materials & Preparation

Read

- **Unit 16**
  Start by reading the unit to learn the content and become familiar with the activities.

- **Mathematical Background**
  Reread the mathematical background carefully to help you think about the important mathematical ideas within the unit. Reread the Student Thinking boxes to anticipate the kinds of ideas and discussions you will likely see during instruction.

- **Angle Measurement Construct Map**
  Read the Angle Measurement Construct Map to help you recognize the mathematical elements in student thinking, and to order these elements in terms of their level of sophistication.

Gather

- ☐ Circular protractors, one for each student
- ☐ Copies of My Protractor worksheet
- ☐ Surveyor tape of different colors
- ☐ Stakes with Flags to mark turning points
- ☐ Yardsticks
- ☐ Clipboards
- ☐ Copies of worksheet--Directions for Walking a Triangle
- ☐ 8.5 × 11 in, blank paper

Prepare

Identify a large enough open space (playground, field, gym) in which students can work to measure distances in yards and use surveyor tape to represent the resulting triangluar path.

Mathematical Background

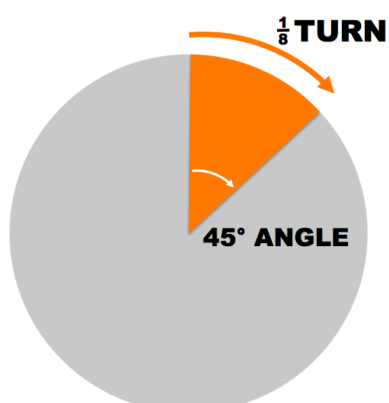
The big ideas of measurement emphasized in this unit include the distinction between units of length measure and units of angle measure. Paths are represented as compositions of straight line segments and turns. The measures of the path help students reason about properties of triangles.

Straight
When is a line straight? Often, when we measure a distance between two points, we imagine a line. A traditional way of thinking about a straight line is as the shortest distance between two points. But it is more consistent with bodily experience to consider a line to be straight when it is formed by moving without any change in direction. For instance, walking at a constant heading while towing a piece of chalk ideally creates a straight line. Or walking toward a landmark while keeping it in one's line of sight creates a straight path.

Angle
An angle is a directed rotation from a heading.

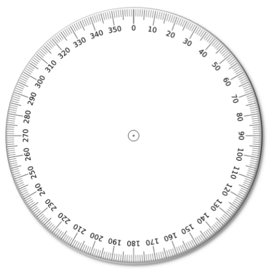
Angle Measure
An angle can be measured as a part of a rotation. One complete rotation or 1 whole turn is the standard. Parts of turns are represented as fractions of a whole turn, as in $\frac{1}{2}$ whole-turn or $\frac{1}{3}$ whole-turn. A degree is $1/360^{th}$ of a whole turn, so 1 whole turn is 360 degrees.

Angles can also be measured as ratios of lengths, as follows. In the diagram below, a $\frac{1}{8}$ whole turn to the right is represented by the arc of the circle. This is a 45 degree clockwise rotation. The ratio of the length of the arc to the circumference of the circle × 360 degrees is 45 degrees. Try it for other degree measures with your own circles.

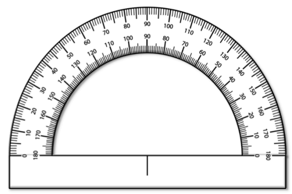


Mathematical Background

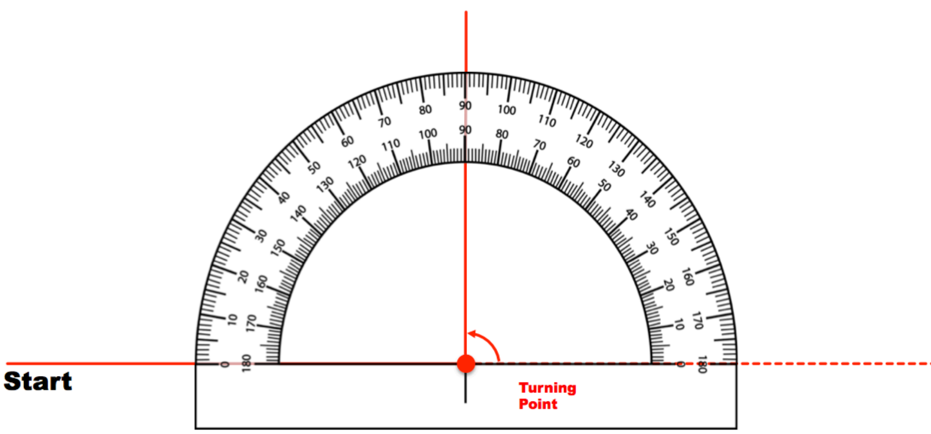
Protractor
A protractor partitions a circle into degrees and these partitions are used to measure angles. A protractor can be used to measure an exterior angle as depicted in the diagram below by extending the line and then measuring the exterior angle to-be-turned.



Circle Protractor



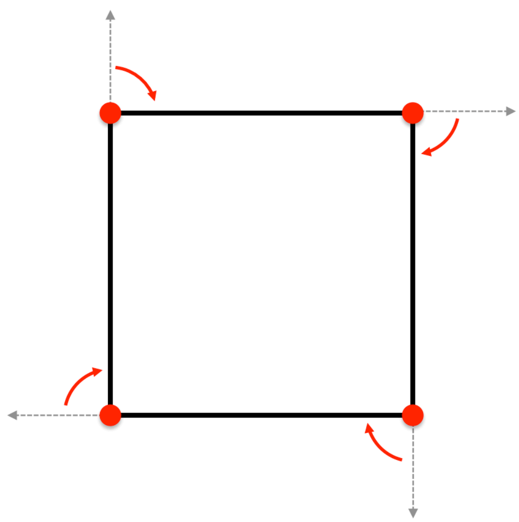
Half Circle Protractor



Mathematical Background

Polygon

A polygon in the plane can be thought of as a path consisting only of straight segments joined by turns, where the person walking the path of the polygon begins and ends at the same place and is oriented in the same heading at the beginning and end of the walk. The path does not intersect itself. The turn angles of a polygon are called the *exterior angles*, and the sum of these turn angles is 1 whole turn. A square illustrates these properties. Imagine starting at the bottom left vertex facing due north or vertically.

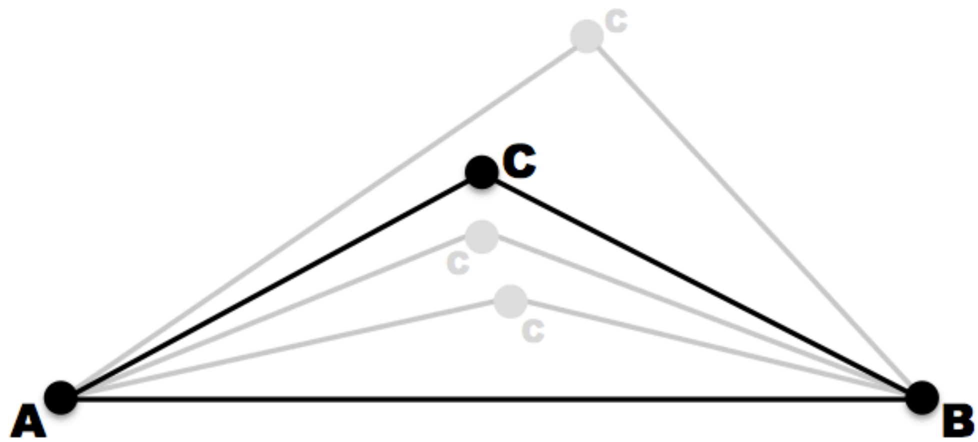


Move straight for a distance, 1u, to produce a length of a side. Then instead of continuing to walk straight, represented by the dashed line, turn to the right $\frac{1}{4}$ of a whole turn. Walk straight again the same distance, 1u, to produce a length of another side of the square. The dashed line represents continuing straight ahead, but instead, turn to the right (clockwise) $\frac{1}{4}$ of a whole turn. Continue in this manner until the square is completed and you are facing in the original direction. The total number of turns is four and total amount of turn is 1 whole-turn (WT):

$$\frac{1}{4}\text{WT} + \frac{1}{4}\text{WT} + \frac{1}{4}\text{WT} + \frac{1}{4}\text{WT} = 1\text{ WT}$$

Triangle

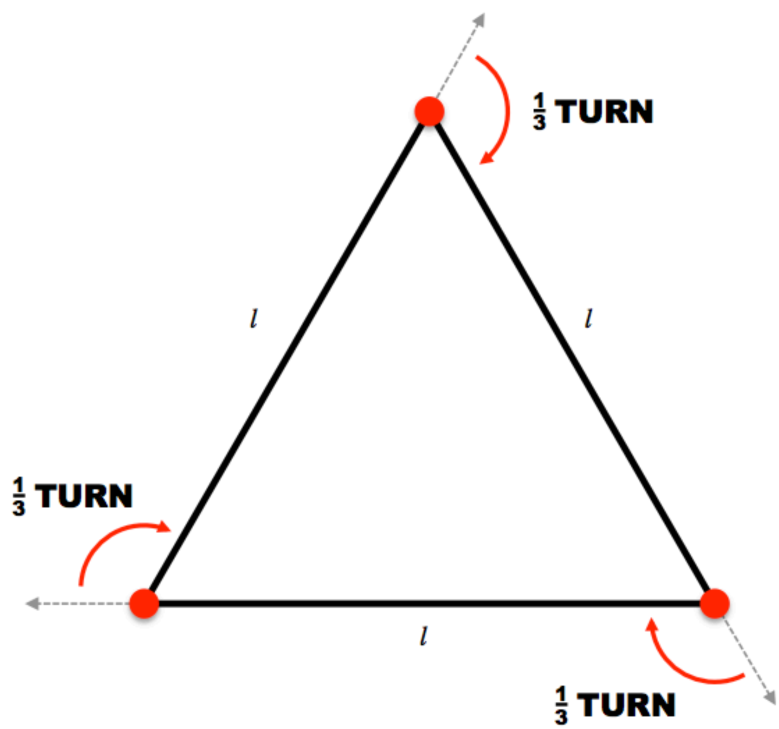
A triangle path begins and ends at the same point, with 3 (non-zero) lengths and 3 (non-zero) turns. The sum of the lengths of any two sides of the triangle must exceed the length of the third side. This is a consequence of straight. For example, in the path below the straight line segment between A and B is also the shortest possible distance between A and B. The path from A to B via C must then be longer than that between A and B. This consequence of straight is called the triangle inequality: The sum of the lengths of any two sides is greater than the length of the third side. For example, AB < AC + BC.



AB < BC + AC

Mathematical Background

An equilateral triangle is constructed by three congruent lengths and turn angles of $\frac{1}{3}$ whole-turn (120 degrees), as shown in the diagram below.



The sum of the exterior angles of the triangle must be 360 degrees, because the path begins and ends at the same position and orientation.

The sum of the interior angles of a triangle must be 180 degrees. At each vertex, the turn angle + the interior angle is 180 degrees because the line segment is straight.

So, we have: $TA_1 + IA_1 = 180$ and $TA_2 + IA_2 = 180$ and $TA_3 + IA_3 = 180$.

So, $TA_1 + IA_1 + TA_2 + IA_2 + TA_3 + IA_3 = 540$.

But $TA_1 + TA_2 + TA_3 = 360$, so $IA_1 + IA_2 + IA_3 = 180$.

Reverse Engineering a Protractor

Whole Group

1. **Today we are going to measure turn-angles.**

   What is a turn angle?

   How do we usually measure turn angles?

   What is meant by 1 whole-turn? Show me 1 whole-turn by turning yourself.

   What is meant by RT $\frac{1}{2}$ whole-turn? Show me by turning yourself.

   Teacher Note

   Elicit student recollections of turn angles developed in previous units. Have students enact whole-turns, $\frac{1}{2}$ turns to the left and to the right.

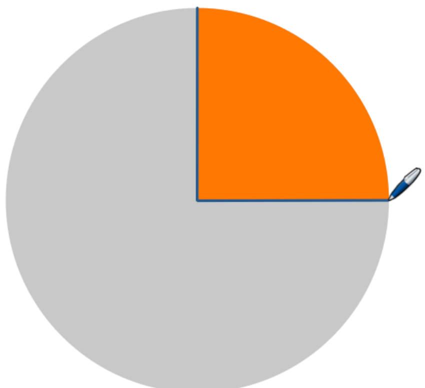
2. **This is a tool used to measure turn-angles. It is called a protractor.** Today we are going to design our own protractor, so we can see better the thinking that led to this one (Show students circular protractor). We will use the circle on the Circle Worksheet to make one of our own.

   Teacher Note

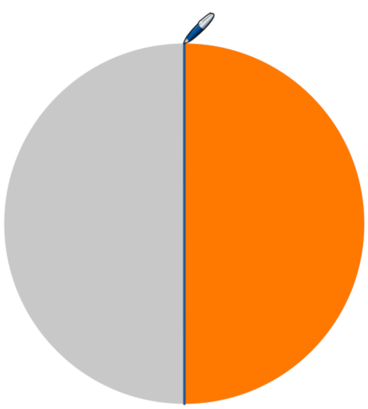
   You may wish to print a copy of the circle worksheet *(page 21)* on a transparency to demonstrate use of the protractor later during this lesson.

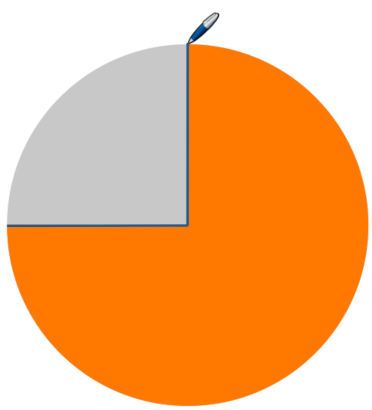
3. **Use your ruler to draw a vertical line segment from the center of the circle to the opposite point on the circumference of the circle.** We will call that the zero point, just as we do for a ruler.

4. **With your partner, talk about how you would represent TR $\frac{1}{4}$ on the circle. Where might you draw a line segment from the center to represent TR $\frac{1}{4}$?** (Teacher elicits and compares student responses and suggests the interpretation depicted below)



5. **With your partner, indicate on your circle TR $\frac{2}{4}$ and TR $\frac{3}{4}$.** (Teacher elicits and compares student responses and suggests the interpretation depicted below)





6. **With your partner, indicate where you should place $\frac{1}{2} \times \frac{1}{4}$ (clockwise) whole-turn.** What is this amount expressed as a fraction? Where would $\frac{5}{8}$ of a whole-turn be?

7. **Where would TR $\frac{1}{3}$ whole-turn be located? $\frac{2}{3}$? $\frac{3}{3}$?**

8. **Where would $\frac{1}{2} \times \frac{1}{3}$ (clockwise) whole-turn be?**

9. **Look at the circular protractor. There are 360 of these small marks. Each mark indicates a fraction of a whole-turn. What is that fraction?** That fraction, 1/360 whole-turn, is called 1 degree. So a turn of 360 degrees is 360 times the rotation of 1 degree. And, 1 degree is 1/360 times the rotation of 1 whole turn.

10. **How many degrees is RT $\frac{1}{4}$ whole-turn? How do you know?** ($\frac{1}{4}$ whole-turn × 360 degrees/whole-turn = 90 degrees).

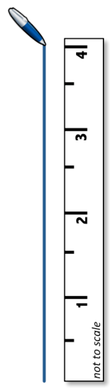
11. **For each of the following, find the degree measure and show it on your circle.**

a. RT $\frac{1}{2}$ whole-turn	**=**	________	**degrees**
b. RT $\frac{2}{4}$ whole-turn	**=**	________	**degrees**
c. RT $\frac{1}{8}$ whole-turn	**=**	________	**degrees**
d. RT $\frac{3}{8}$ whole-turn	**=**	________	**degrees**
e. RT $\frac{6}{8}$ whole-turn	**=**	________	**degrees**
f. RT $\frac{1}{3}$ whole-turn	**=**	________	**degrees**
g. RT $\frac{2}{3}$ whole-turn	**=**	________	**degrees**
h. RT $\frac{1}{2} \times \frac{1}{3}$ whole-turn	**=**	________	**degrees**

12. **Find the degree measures that you have indicated on your circle on the protractor.**

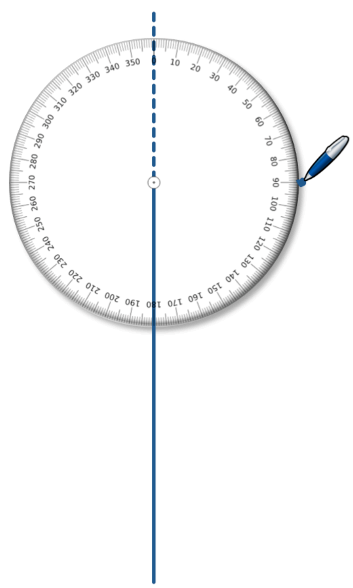
Using the Protractor to Make a Rectangle Path

1. **Let's use the protractor and ruler to draw a rectangle path that is 4 inches by 6 inches.** To get started, draw a straight line on paper that is 4 inches, oriented vertically, like this:

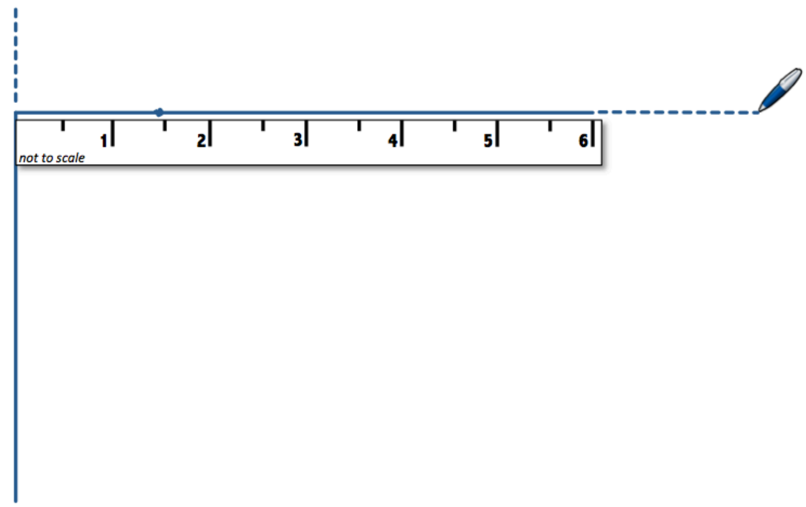


2. **Draw a dotted line to extend the 4 inch side (a).** Center the protractor at this point and mark a 90 degree turn to the right (b).

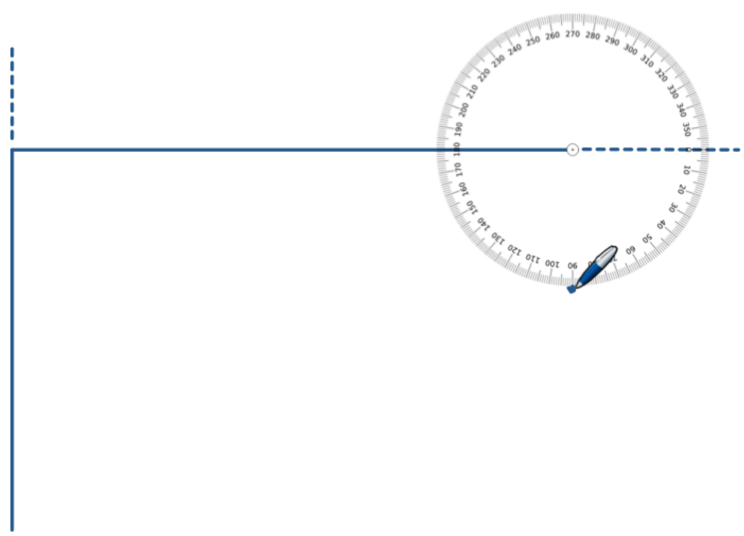
a:


b:


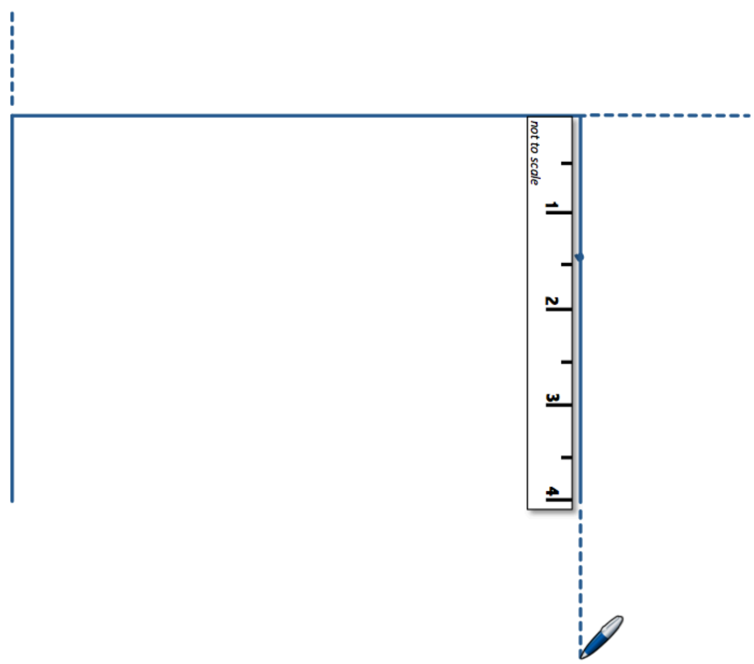
Draw a 6-inch line segment.



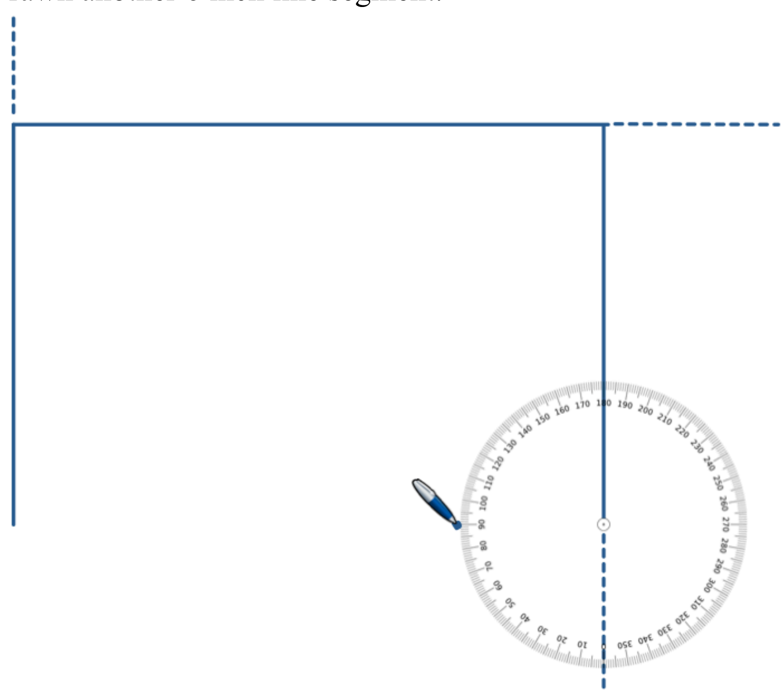
3. **Draw a dotted line to extend the 6 inch side.** Center the protractor at this point and mark a 90 degree turn to the right.



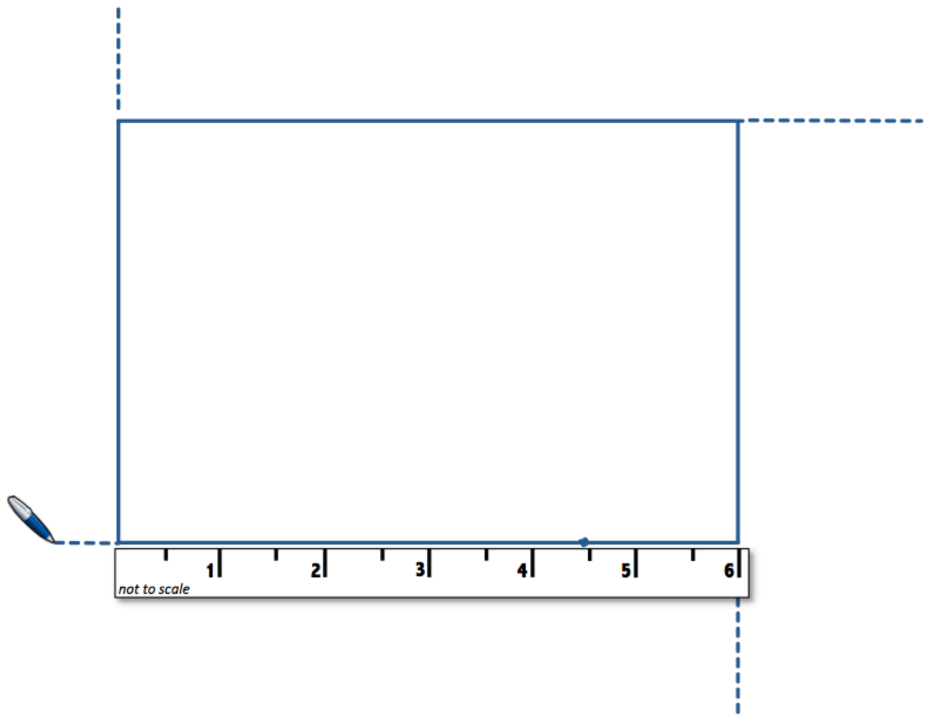
Draw another 4 inch line segment.



4. **Extend the 4 inch line segment.** Mark a 90 degree turn to the right. Drawn another 6 inch line segment.



5. **Extend the 6 inch side.** Mark a 90 degree turn to the right. It should bring the bring the extended segment (the dotted line) onto the existing 4 inch side.



Making an Equilateral Triangle

Partners

1. **Draw a path with your ruler and protractor that makes a triangle with all sides the same length.** Use dotted lines at the turning points, so that it is easy to see the turn angles. Then write directions so that someone else could reproduce the same path on the Triangle Ribbon Path Worksheet.

Whole Group

2. **Compare student solutions.**

Teacher Note
Emphasize the use of extended line segments and the need to turn 120 degrees at each vertex. A possible extension is to ask what is the sum of the turn angles? Why do you think it is 360 degrees? (Be sure to include the third turn angle)

Partner

3. **What is each interior angle in the equilateral triangle? How could you find it without measuring?**

4. **What is the sum of the interior angles in an equilateral triangle?**

Whole Group

5. **Compare student solutions with an eye toward using properties of straight (a straight line has a measure of 180 degrees at any point).**

Making Paths for any Triangle

Partner

1. Use your ruler to draw any triangle you like that is not equilateral.
2. Use your ruler and protractor to write directions for making the triangle you have drawn.
3. What is the sum of the turn angles for your triangle?
4. What is the sum of the interior angles for your triangle?
5. For your triangle, what do you notice about the sum of any two side lengths compared to the third length? Why do you think this is happening?
6. How would you define a triangle so that everyone could tell a triangle from some other figure, like a rectangle?

Whole Group

Compare selected constructions.

Teacher Note

During the whole-class conversation, the goal is to help students understand that the sum of the exterior or turn angles must be 360 degrees, and that sums that are not 360 degrees reflect measurement error. If 1 whole turn has a measure of 360 degrees, then any closed, non-intersecting path, like that of a triangle, must be produced by one whole turn. So, it is a necessity, not an empirical approximation, that the sum of the turn angles is 360 degrees.

If the sum of the exterior angles is 360 degrees, then the sum of the interior angles of a triangle must be 180 degrees. See the mathematical background.

Challenge student definitions to promote more precision. For example, definitions such as 3 sides, 3 turns can be challenged by 3 segments and 3 turns that are not closed, as in Z.

Formative Assessment

Give the formative assessment. For the first item, select responses that provide an opportunity for students to explain their choices of length and angle measures. For the second item, select responses that provide an opportunity for students to explain their solution strategy, especially solutions that take advantage of the sum of meaures of the turn angles as 360 degrees. Focus as well on student explanations of the necessity of the interior angle sum as 180 and the triangle inequality.

Formative Assessment

Name: ________________________Date:________________

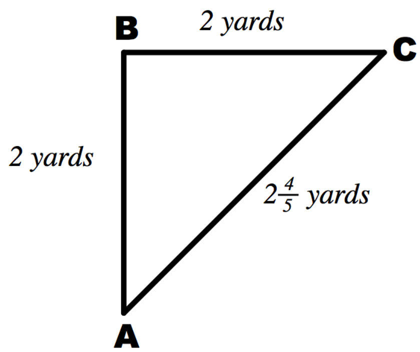
1. After selecting a starting point, complete these directions to make an equilateral triangle.

Walk straight 6 yards. Turn Right _______ degrees.

Walk straight _____ yards. Turn ___________________.

Walk straight ____ yards. Turn ____________________.

2. Here is a triangle, ABC, with lengths in yards.



Complete these directions to construct this triangle.

Directions:

At **A,** look straight ahead.
Walk 2 yards; turn __________________
Walk _______; turn Right 135 degrees.
Walk _______; turn __________________

What is the sum of the turn angles of the triangle?
___________degrees; explain why.

What is the sum of the interior angles?
__________ degrees; explain why.

Why is the distance AC less than the sum of the distances between AB and BC?

Formative Assessment Record

Ribbon Paths **Unit 16**

Student ___________________________ Date __________________

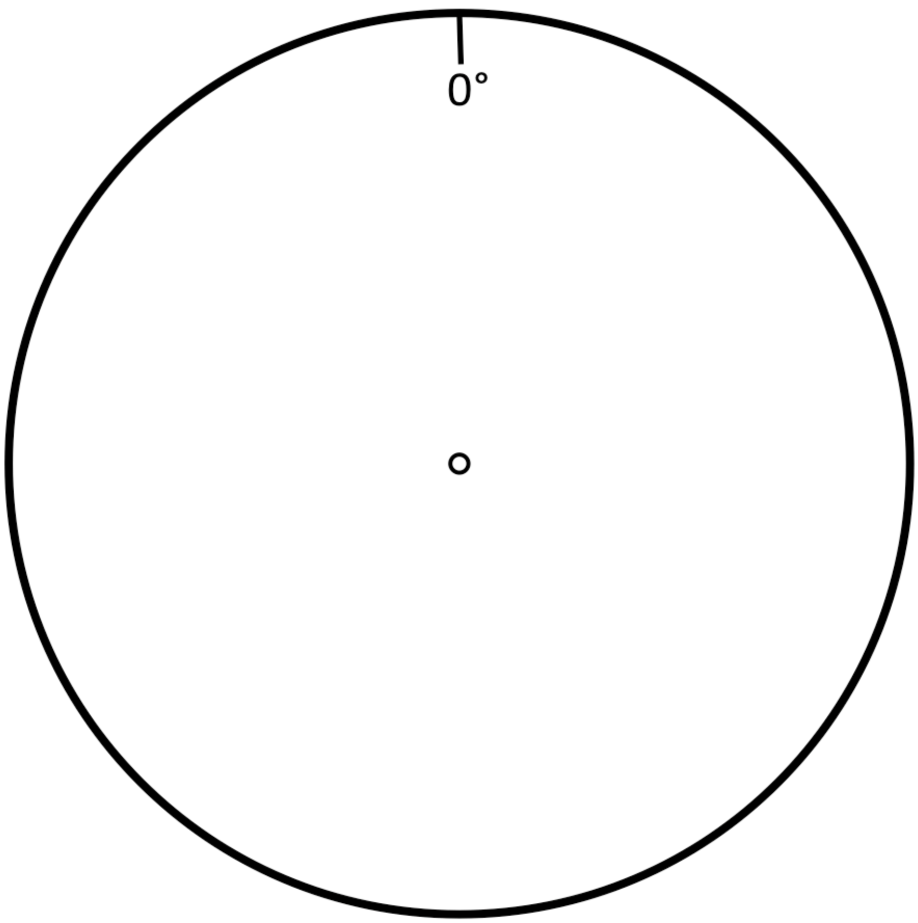
Indicate the levels of mastery demonstrated in students' work outdoors and in their responses to the formative assessment items by circling those for which there is clear evidence:

Level	Description	Notes
Item 1: Figure-as-path	+ Completes equilateral triangle directions correctly, indicating differentiation between length and angle measures. 0 Response indicates failure to differentiate turns from lengths, or does not recall that 1/3 turn is 120 degrees.	
Item 2: Properties of Triangles	+ Explains the basis of the triangle inequality, the interior and exterior angle sums and uses knowledge of the 360 exterior angle sum to create valid path directions. 0 Explains the basis of two properties (interior sum, exterior angle sum, inequality of lengths) of triangles. Uses knowledge of 360 exterior angle sum to create valid path directions. - Other partial knowledge of triangles.	
ToMA^{0}3A	Recognize and compare amount of turns. Performance: Student successfully incorporates magnitude and direction of turns into both paths, so is correct for both items.	
ToMA^{0}2B	Compare angles as turns. Performance: Student successfully incorporates turns into paths but may not correctly quantify each turn measure.	

Academic Language: Indicate academic words the student is familiar with by recording them here.

Circle Worksheet

Print on Trancparency Paper



Triangle Ribbon Path Worksheet

Name: ________________________Date:________________

Directions for a Triangle Ribbon Path

START FACE____________________

WALK STRAIGHT ________________TURN_______________

WALK STRAIGHT ________________TURN_______________

WALK STRAIGHT ________________TURN_______________

STOP

Diagram of Path