Making Sense of Multiplying Fractions

UNIT

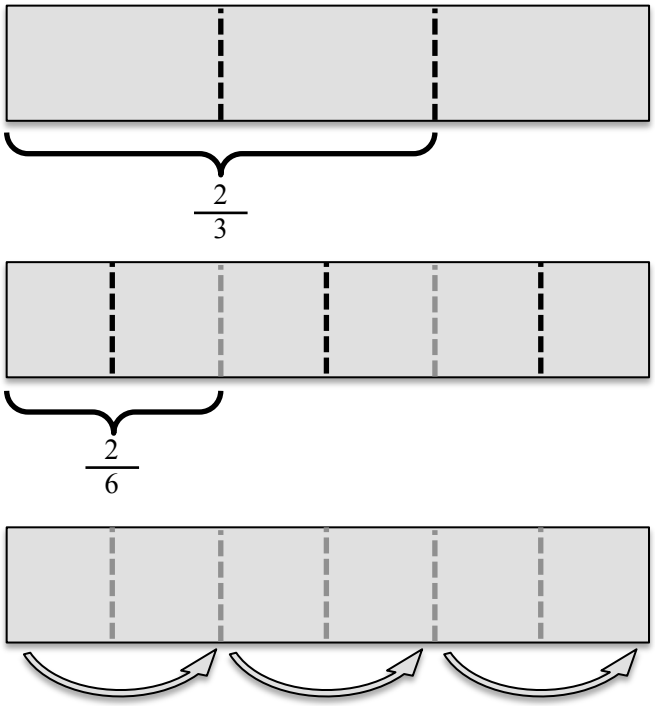
17

Contents

Mathematical Concepts
Unit Overview
Mathematical Background
Instruction
Multiplying a Composite Unit Length by a Unit Fraction
Multiplying a Unit Length by a Unit Fraction
Multiplying a Unit Length by a Non-Unit Fraction
Multiplying a Non-Unit Length by a Unit Fraction
Multiplying a Non-Unit Length by a Non-Unit Fraction
Formative Assessment
Formative Assessment Record

Mathematical Concepts

- $\frac{a}{b} \times c$ unit (c =1 or $c > 1$) means b-splitting the c unit and then iterating that split a times.
  - $\frac{3}{2} \times 4u = 6u$ means 2-split $4u$ ($\frac{1}{2}$ of $4u$) resulting in $2u$ and iterate $2u$ 3 times for a result of $6u$.
- $\frac{a}{b} \times \frac{c}{d}$ unit (<1) means b-splitting $\frac{c}{d}$ and then iterating that bd-split a times. Symbolically, a $\times$ ($\frac{1}{b} \times \frac{c}{d}$) unit. For example,
  - $\frac{3}{2} \times \frac{2}{3} u = \frac{6}{6} u$ means:



$\frac{2}{3}$ unit – length

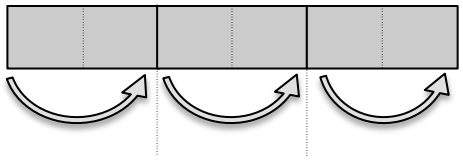
$\frac{1}{2}$ of the $\frac{2}{3}$ unit (the 2-split) resulting in $\frac{2}{6}$ unit.

3 copies of $\frac{2}{6}\ unit = \frac{6}{6}\ unit$

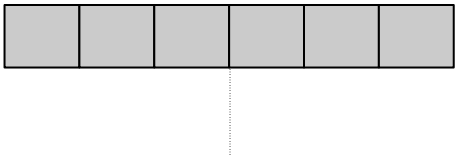
- $\frac{a}{b} \times c$ unit (c =1 or $c > 1$) also means a iterations of the c unit, followed by b-splitting ac.
  - $\frac{3}{2} \times 4u = 6u$ means 3 copies of $4u$, resulting in $12u$, and then 2-split $12u$ for a result of $6u$.

- $\frac{a}{b} \times \frac{c}{d}$ unit (<1) means *a* copies of $\frac{c}{d}$ and then *b* splitting $\frac{ac}{d}$. Symbolically, $\frac{1}{b}x\ (\ a \times \frac{c}{d}\ )$ unit. For example, $\frac{3}{2} \times \frac{2}{3}\ nit = \frac{6}{6}$ *nit is* $\frac{3}{2} \times \frac{2}{3} nit = \frac{6}{6} nit.$

$\frac{2}{3}\ nit$



3 copies of $\frac{2}{3}\ nit = \frac{6}{3} nit$



2-split of $\frac{6}{3}\ nit = \frac{3}{3} nit\ =\ \frac{6}{6}\ nit\ u$

- An algorithm for multiplication that is derived from this sense of splitting and iterating is: $\frac{a}{b} \times \frac{c}{d} = \frac{a \text{ x } c}{b \text{ x } d}$

Materials & Preparation

Mathematical Concepts
Unit Overview
Materials and Preparation
Mathematical Background
Multiplying a Composite Unit
Length by a Unit Fraction
Multiplying A Unit
Length by a Unit Fraction
Multiplying a Unit
Length by a Non-Unit Fraction
Multiplying a Non-Unit
Length by a Unit Fraction
Multiplying a Non-Unit
Length by a Non-Unit Fraction
Formative Assessment

Unit Overview

Using paper strip units, students first re-visit multiplication of composite units (e.g., 2 u) by unit ($\frac{1}{b}$) fractions. Next, they investigate the meaning of multiplying a unit fraction length (e.g. $\frac{1}{2}\,u$) by a unit fraction (e.g. $\frac{1}{3}$). The third investigation explores multiplication of a non-unit fraction length ($\frac{2}{4}$ unit) by a unit fraction (e.g. $\frac{1}{2}$). The fourth investigation explores multiplication of non-unit fraction length by a non-unit fraction, such as $\frac{2}{3}$. A multiplication algorithm is introduced to generalize reliably across these different situations.

Read

- ☐ **Unit 17**
  Start by reading the unit to learn the content and become familiar with the activities. Try out all the investigations yourself, before teaching.

- ☐ **Mathematical Background**
  Reread the mathematical background carefully to help you think about the important mathematical ideas within the unit.

Prepare

- ☐ Provide students with paper strips to represent units of measure.

Mathematical Concepts
Unit Overview
Materials and Preparation
Mathematical Background
Multiplying a Composite Unit
Length by a Unit Fraction
Multiplying A Unit
Length by a Unit Fraction
Multiplying a Unit
Length by a Non-Unit Fraction
Multiplying a Non-Unit
Length by a Unit Fraction
Multiplying a Non-Unit
Length by a Non-Unit Fraction
Formative Assessment

Mathematical Background

The algorithm $\frac{a}{b} \times \frac{c}{d} = a \times \frac{1}{b} \times \frac{c}{d}$ compresses the steps taken with paper strip folding but really does the same thing to the unit-strip. Here are some cases that help visualize the action of the algorithm:

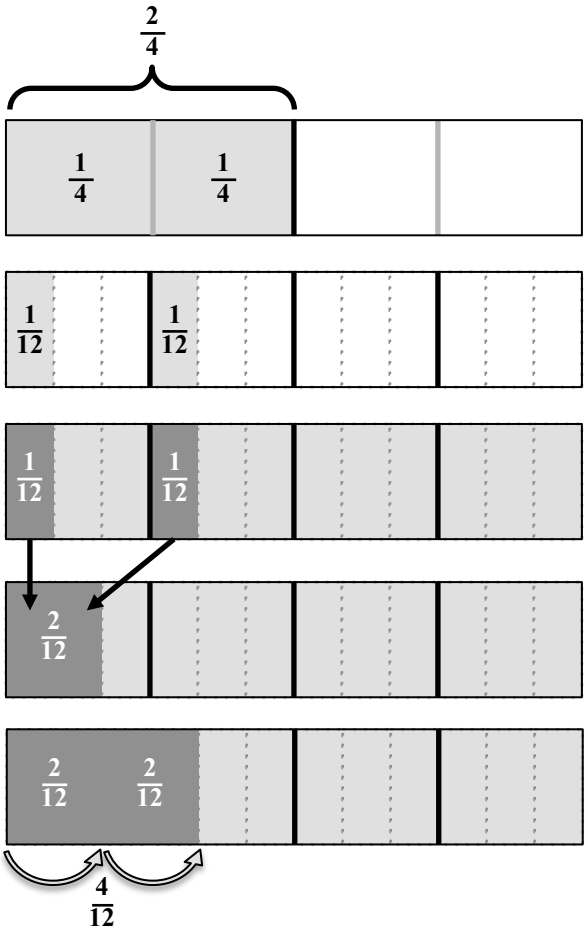
Multiplying Denominators: If we found $\frac{1}{3} \times \frac{1}{2}$ unit, we begin with the folded $\frac{1}{2}$ unit. The notation $\frac{1}{2}$ unit hides its meaning, which is $\frac{1}{2}$ x 1 unit. We represent this splitting action compactly as $\frac{1}{2}$ unit. Multiplying by $\frac{1}{3}$ means splitting the $\frac{1}{2}$ unit into 3 congruent partitions. Then each partition is $\frac{1}{6}$ unit long (we can unfold the paper and check). 2 and 3 are factors of 6, so to find the total number of partitions we multiply the denominators. Multiplying denominators is a short-cut: If we folded a unit strip, first we would have to split the unit into 2 congruent partitions (the $\frac{1}{2}$ unit), then each of these partitions would be split into 3 congruent partitions, for a total of 6 congruent partitions. The algorithm makes this a one-step process.

Multiplying Numerators: The meaning of the numerator is to iterate the split-unit that many times. For the $\frac{1}{3} \times \frac{1}{2}$ unit, the 1 in $\frac{1}{3}$ suggests 1iteration of the split-unit or $\frac{1}{6}$ unit. The algorithm obtains the same result. If instead, we found $\frac{2}{3} \times \frac{1}{2}$ unit, we know that we need to iterate the $\frac{1}{6}$ split-unit two times.

Mathematical Background

Mathematical Concepts
Unit Overview
Materials and Preparation
Mathematical Background
Multiplying a Composite Unit
Length by a Unit Fraction
Multiplying A Unit
Length by a Unit Fraction
Multiplying a Unit
Length by a Non-Unit Fraction
Multiplying a Non-Unit
Length by a Unit Fraction
Multiplying a Non-Unit
Length by a Non-Unit Fraction
Formative Assessment

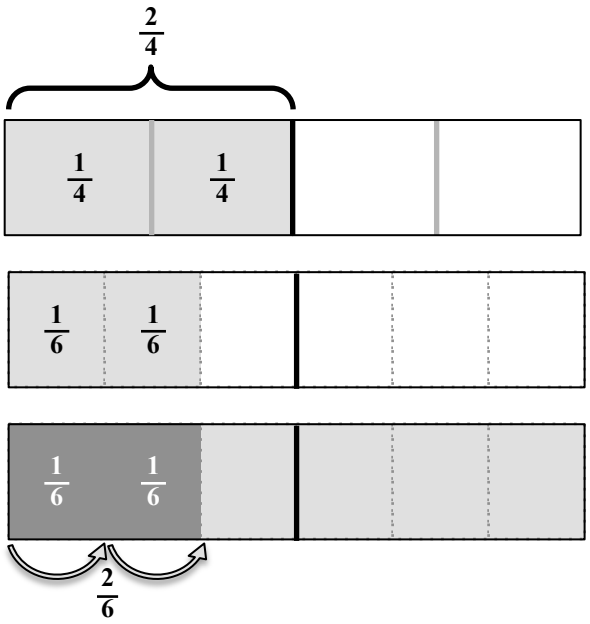
Non-unit numerators and denominators: If neither the unit-length nor the multiplier is expressed as a unit fraction, the rationale is the same. For example, the algorithm suggests that the result of $\frac{2}{3} \times \frac{2}{4}$ unit is $\frac{4}{12}$ unit. Considering paper folding, the denominator of $\frac{2}{3}$ acts on $\frac{2}{4}$ unit by splitting each $\frac{1}{4}$ unit into 3 congruent partitions, each of which is $\frac{1}{12}$ of the original unit. There are 2 of these $\frac{1}{12}$ units, for a total of $\frac{2}{12}$ units. The numerator of the $\frac{2}{3}$ suggests 2 iterations of the $\frac{2}{12}$ unit, resulting in $\frac{4}{12}$ unit. The algorithm compresses each of these steps.



Mathematical Background

Mathematical Concepts
Unit Overview
Materials and Preparation
Mathematical Background
Multiplying a Composite Unit
Length by a Unit Fraction
Multiplying A Unit
Length by a Unit Fraction
Multiplying a Unit
Length by a Non-Unit Fraction
Multiplying a Non-Unit
Length by a Unit Fraction
Multiplying a Non-Unit
Length by a Non-Unit Fraction
Formative Assessment

Another way to think about this operation is that $\frac{2}{3}$ x $\frac{2}{4}$ unit means to partition the $\frac{2}{4}$ unit into 3 congruent partitions, each of which is $\frac{1}{6}$ or $\frac{2}{12}$ of the original unit. Then iterate the $\frac{1}{6}$ unit 2 times, for a total of $\frac{2}{6}$ units which is equivalent to $\frac{4}{12}$ unit length. Both ways of thinking about multiplication are illustrated in the diagrams that follow.



As we noted earlier with mathematical concepts, another way of thinking about $\frac{2}{3}$ x $\frac{2}{4}$ unit is to first iterate $\frac{2}{4}$ unit 2 times, and then to 3-split the resulting $\frac{4}{4}$ unit, which results in $\frac{1}{3}$ unit. The algorithm produces an equivalent result, $\frac{4}{12}$ unit. Algebraically, $\frac{1}{3}$ x 2 x $\frac{2}{4}$ unit.

Mathematical Concepts
Unit Overview
Materials and Preparation
Mathematical Background
Multiplying a Composite Unit Length by a Unit Fraction
Multiplying A Unit Length by a Unit Fraction
Multiplying a Unit Length by a Non-Unit Fraction
Multiplying a Non-Unit Length by a Unit Fraction
Multiplying a Non-Unit Length by a Non-Unit Fraction
Formative Assessment

Multiplying a Composite Unit Length by a Unit Fraction

Whole Group / Individual

1. **The teacher holds a strip that is 10 cm long and asks all students to:**

    a. Show me, by folding this 10 cm strip, the result of $\frac{1}{2} \times 10$ cm.

    Q: What did you do? (2 split the strip)
    Q: What was the result? (5 cm)
    Q: How could you demonstrate that it must be 5 cm. and not something else, like 6 cm (iterate it 2 × to establish that 10 cm is 2 times as long as 5 cm)
    Q: 10 cm is ____ times as long as 5 cm.
    Q: 5 cm is ____ times as long as 10 cm.

    b. Predict the result of $\frac{1}{10} \times 10$ cm.

    Q: How could you demonstrate that the result must be 1 cm?
    Q: 10 cm is ____times as long as 1 cm.
    Q: 1 cm is ____ times as long as 10 cm.

Mathematical Concepts
Unit Overview
Materials and Preparation
Mathematical Background
Multiplying a Composite Unit
Length by a Unit Fraction
Multiplying A Unit
Length by a Unit Fraction
Multiplying a Unit
Length by a Non-Unit Fraction
Multiplying a Non-Unit
Length by a Unit Fraction
Multiplying a Non-Unit
Length by a Non-Unit Fraction
Formative Assessment

Multiplying a Unit Length by a Unit Fraction

Partners

1. **Here is a 1 foot long unit. With a partner, try out each of these:**

   a. $\frac{1}{4} \times 1\ ft =$ _____

   b. $\frac{1}{3} \times 1\ ft =$ _____

   c. $\frac{1}{2} \times \frac{1}{3} \times 1\ ft =$ _____

2. **The teacher leads a discussion of the solution for each problem. Productive questions include:**

   Q: How can you check on your answer? (Iterate the $\frac{1}{b}$, b times to reproduce the unit length)
   Q: 1 *ft* is ____times as long as ____ (result)
   Q: (Result) is ____times as long as 1 *ft*.

Mathematical Concepts
Unit Overview
Materials and Preparation
Mathematical Background
Multiplying a Composite Unit
Length by a Unit Fraction
Multiplying A Unit
Length by a Unit Fraction
Multiplying a Unit
Length by a Non-Unit Fraction
Multiplying a Non-Unit
Length by a Unit Fraction
Multiplying a Non-Unit
Length by a Non-Unit Fraction
Formative Assessment

Multiplying a Unit Length by a Non-Unit Fraction

Partners

1. **Here is the 1 foot long unit again. With a partner, try out each of these:**

   a. $\frac{2}{4} \times 1\,ft =$ _____

   b. $\frac{2}{3} \times 1\,ft =$ _____

   c. $\frac{3}{2} \times 1\,ft =$ _____

   d. $\frac{3}{3} \times 1\,ft =$ _____

Whole Group

2. **The teacher leads a discussion of the solution for each problem. Productive questions include:**

   Q: What does the bottom number tell us? (How many equal partitions of the foot?)
   Q: What does the top number tell us? (How many copies of the equal partitions)
   Q: Looking at these, for which would you predict a result shorter than 1 foot? Longer than 1 foot? The same length as 1 foot? Why do you think so?
   Q: The (product unit length) is ____ times as long as 1 *ft*.
   Q: 1 ft. is ____ times as long as (product unit length)

Mathematical Concepts
Unit Overview
Materials and Preparation
Mathematical Background
Multiplying a Composite Unit Length by a Unit Fraction
Multiplying A Unit Length by a Unit Fraction
Multiplying a Unit Length by a Non-Unit Fraction
Multiplying a Non-Unit Length by a Unit Fraction
Multiplying a Non-Unit Length by a Non-Unit Fraction
Formative Assessment

Multiplying a Non-Unit Length by a Unit Fraction

Partners

1. **Here is a $\frac{1}{2}$ foot long unit. With a partner, try out each of these:**

   a. $\frac{1}{4} \times \frac{1}{2} ft =$ _____

   b. $\frac{1}{3} \times \frac{1}{2} ft =$ _____

   c. $\frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} ft =$ _____

Whole Group

2. **The teacher leads a discussion of the solution for each problem. Productive questions include:**

   Q: How can you check on your answer? (iterate the $\frac{1}{b}$ result $\frac{b}{2}$ times to reproduce the unit length)
   Q: $\frac{1}{2} ft$ is _____times as long as ___(the product unit length)
   Q: (The product unit length) is ____ times as long as $\frac{1}{2}$ ft.

Mathematical Concepts
Unit Overview
Materials and Preparation
Mathematical Background
Multiplying a Composite Unit Length by a Unit Fraction
Multiplying A Unit Length by a Unit Fraction
Multiplying a Unit Length by a Non-Unit Fraction
Multiplying a Non-Unit Length by a Non-Unit Fraction
Formative Assessment

Multiplying a Non-Unit Length by a Non-Unit Fraction

Whole Group

1. **Let try this one together. $\frac{2}{4} \times \frac{1}{2}$ Obama. Let's use an Obama because even $\frac{1}{2}$ of it is long enough for everyone to see.**

   Take a 2 ft strip of paper, call it 1 OB. Ask students how to find $\frac{1}{2}$ of it (Fold into 2 equal partitions, establish by iteration that the partitions must be the same length as 2 copies are just as long as 1 OB.)

   Tape the 1 OB and the $\frac{1}{2}$ OB to the board so that everyone can see them.

   Hold another copy of $\frac{1}{2}$ OB and ask: What does it mean to multiply this $\frac{1}{2}$ OB by $\frac{2}{4}$? What is the bottom number telling us? What is the top number telling us?

   Let's split the $\frac{1}{2}$ OB into 4 equal parts. How can we do that ($\frac{1}{2} \times \frac{1}{2}$)?

   Now that we have split the $\frac{1}{2}$ OB into 4 equal parts, what do I call each part, if I think of 1 OB? ($\frac{1}{8}$) If I am right and this is $\frac{1}{8}$, how many copies will it take to make 1 OB? (Enact 8 iterations on the taped copy of 1 OB.

   Unfold the strip so that students can see the 8th's. What does the top number tell us to do? (Travel 2 copies). Enact traveling 2 copies on the 1 OB. How far is that all together? ($\frac{2}{8}$ OB or $\frac{1}{4}$ OB).

2. **If we wanted to make a method so that people would not have to fold paper to find out that $\frac{2}{4} \times \frac{1}{2}$ OB = $\frac{2}{8}$ OB, what might we try?**

   $2 \times \frac{1}{4} \times 2$ Multiply the numerators and multiply the denominators.

Mathematical Concepts
Unit Overview
Materials and Preparation
Mathematical Background
Multiplying a Composite Unit
Length by a Unit Fraction
Multiplying A Unit
Length by a Unit Fraction
Multiplying a Unit
Length by a Non-Unit Fraction
Multiplying a Non-Unit
Length by a Non-Unit Fraction
Multiplying a Non-Unit
Length by a Non-Unit Fraction
Formative Assessment

Partner

3. **With your partner, try out our new method for multiplying fractions, find the result, and then see if you can figure out whether or not the result is right with the paper strips.**

   a. $\frac{2}{4} \times \frac{2}{4}$ BF = _____

   b. $\frac{2}{4} \times \frac{2}{4}$ BF = _____

   c. $\frac{3}{4} \times 2$ BF (We can write this as $\frac{3}{4} \times \frac{2}{1}$ BF)

Whole Group

4. **Class discussion using split and copy to justify the product algorithm. Be sure to use BF strips to show explicitly each split and copy.**

Formative Assessment

Mathematical Concepts
Unit Overview
Materials and Preparation
Mathematical Background
Multiplying a Composite Unit
Length by a Unit Fraction
Multiplying A Unit
Length by a Unit Fraction
Multiplying a Unit
Length by a Non-Unit Fraction
Multiplying a Non-Unit
Length by a Unit Fraction
Multiplying a Non-Unit
Length by a Non-Unit Fraction
Formative Assessment

1. Find the product of:

   a. $\frac{1}{2} \times \frac{1}{4}$ unit = _____

   b. $\frac{2}{4} \times \frac{1}{4}$ unit = _____

   c. $\frac{2}{4} \times \frac{2}{4}$ unit = _____

   d. $\frac{2}{3} \times 2$ unit = _____

   e. $\frac{4}{5} \times 5$ unit = _____

   f. $\frac{3}{10} \times \frac{10}{3}$ unit = _____

2. Use the BF units to show why $\frac{1}{2} \times \frac{1}{2}$ BF = $\frac{1}{4}$ BF:

3. Use the BF units to show why $\frac{2}{4} \times \frac{1}{2}$ BF = $\frac{2}{8}$ BF:

4. Use the BF units to show why $\frac{3}{2} \times \frac{2}{4}$ BF = $\frac{6}{8}$ BF:

Formative Assessment Record Making Sense of Multiplying Fractions Unit 17

Student _____ Date _____

For each student, indicate

Level	Description	Notes
Interprets $\frac{a}{b} \times \frac{c}{d}$-unit	Gets most items right and responds appropriately to 2,3, 4	
Understands $\frac{a}{b} \times \frac{c}{d}$-unit for non-unit choice of $\frac{a}{b}$	Items 1, 2 correct result.	
Understands $\frac{a}{b} \times \frac{c}{d}$-unit only for unit fractions choice of $\frac{a}{b}$ and $\frac{c}{d}$	Item 1 correct result. Item 2 correct drawing but does not describe relation as $\frac{3}{2}$ in number sentence.	
Does not interpret multiplication with reference to splits and copies	Correct response to item 1.	
Procedural Competence with Multiplication of $\frac{a}{b} \times \frac{c}{d}$	Gets most procedural items correct.	
Emerging Procedural Competence	Correct responses on a few items.	
NL	Cannot interpret the meaning of fraction multiplication and cannot reliably execute appropriate procedures.	

Academic Language: Indicate academic words the student is familiar with by recording them here.