Measuring Parts of Units, Fraction Arithmetic with 1/4

Mathematical Concepts

- Fractions represent partial units.
- 1/b represents 1 copy of a unit partitioned into *b* congruent (equi-partitioned) parts.
- a/b represents *a* copies of a unit partitioned into *b* congruent parts.
- a/b can be interpreted as traveling from zero to the location a/b.
- a/b can also be interpreted as iterating a length of 1/b, a times.

Unit Overview

The lesson begins by composing 2 successive 2-splits of 1 ft., symbolizing the action as multiplication (½ of ½ of 1 ft.). The iterative property of equal partitions, n of 1/n units re-creates 1 unit, is employed to establish that 4 iterations of ¼ ft. are congruent with 1 ft. The bottom number of the fraction is interpreted as the number of equal parts (4) and the top number as the number of copies of one of these equal parts. Students create a 2-unit strip with each unit split into fourths. Students then "travel" *a* copies of 4, where *a* ranges from 1 to 9. Travel is related to addition and the split-copy to multiplication. Subtraction is modeled by the difference between lengths. The unit concludes by locating ¼ ft. on every unit of the tape measure constructed during the previous unit, labeling the accumulation of ¼ ft. units from the origin; establishing equivalence (same location) among eighths, fourths and halves; and using the tape measure to measure lengths of snakes that are not whole number units long. The formative assessment includes items ordering fractional unit lengths, locating ¼, $^{2}/_{4}$ or ½, and $^{5}/_{4}$ on a number line, and enacting the meaning of multiplication of 1 unit by a 4-split fraction. The formative assessment also provides insight about how students are reasoning about the meaning of addition and subtraction of fractional quantities.

Unit

9

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Materials & Preparation

Read

- ☐ **Unit 9**
  Start by reading the unit to learn the content and become familiar with the activities.
- ☐ **Mathematical Background**
  Reread the Mathematical Background to anticipate the kinds of ideas and discussions you will likely see during instruction.
- ☐ **Measurement Construct Map**
  Read the construct map and look at the multimedia map to help you recognize the mathematical elements in student thinking, and to order these elements in terms of their level of sophistication (ToM, composing 2-splits).

Gather

- ☐ Student math journals
- ☐ Teacher journal for note-taking
- ☐ Paper for personal unit strips
- ☐ Glue or tape
- ☐ Markers for labeling tape measures
- ☐ Chart paper, markers
- ☐ Several items to measure
- ☐ Footstrip rulers from previous units for further annotation

Prepare

- ☐ 10 personal unit strips for each student:
  5 – for creating a tape measure
  5 – for testing ideas

  Students should receive different sizes of personal unit: 1 x 4, 1 x 8, or 1 x 16. More proficient students should use the shortest units *(see page 5)*.

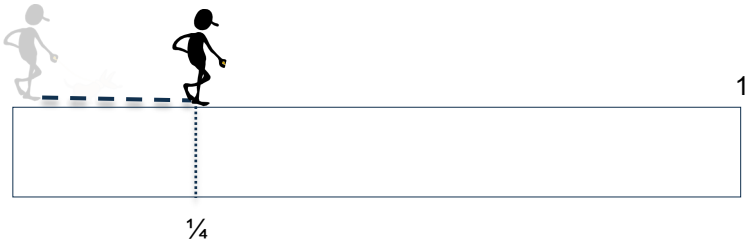
- ☐ 10 to 15 personal unit strips for teacher demonstration: teacher unit should be a 2-foot strip.

Mathematical Background

Fraction Arithmetic (1/4) Unit 9

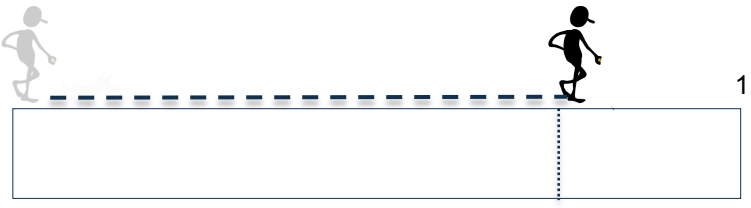
Two metaphors are used to help students understand the meaning of a fractional quantity. One relies on unit iteration where, for example, $^3/_4$ means 3 iterations of ¼ unit. This sense of fraction relies on a copy of a partitioned unit, as in 3 copies (iterations) of ¼ unit results in $^3/_4$ unit. The second metaphor is distance traveled where, for example, $^3/_4$ means starting at 0 and moving $^3/_4$ unit. Both of these metaphors, one static and the other dynamic, help students form images of fractions that will later help them locate fractions on the number-line, because the number-line is an idealized ruler.

Distance traveled – moving ¼ unit:

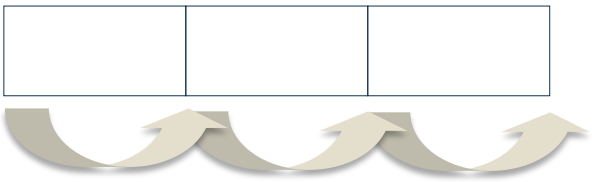


¼ unit iterated once:

Distance traveled – moving $^3/_4$ unit:

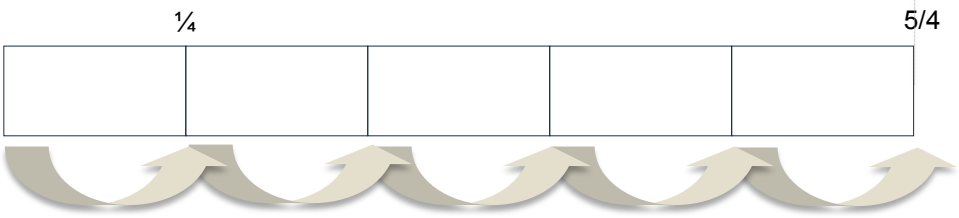


¼ unit iterated three times:



Distance traveled – moving $^5/_4$ unit:

¼ unit iterated five times:

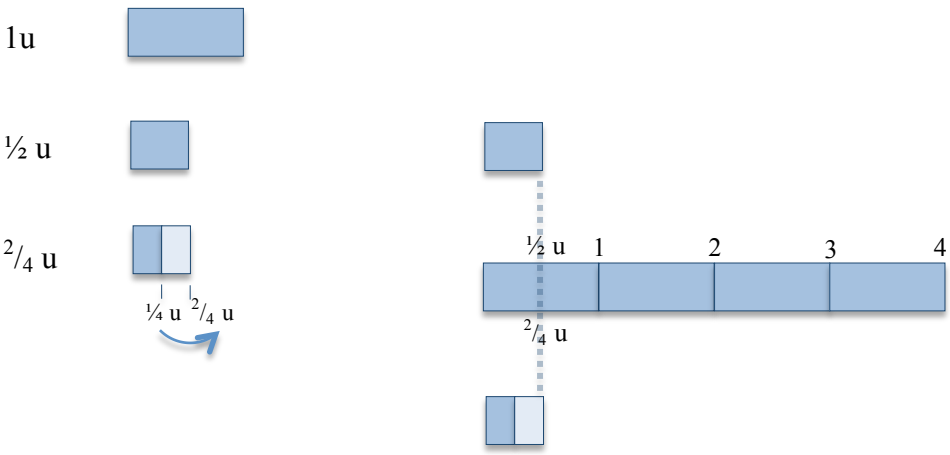


Mathematical Background

Equivalence

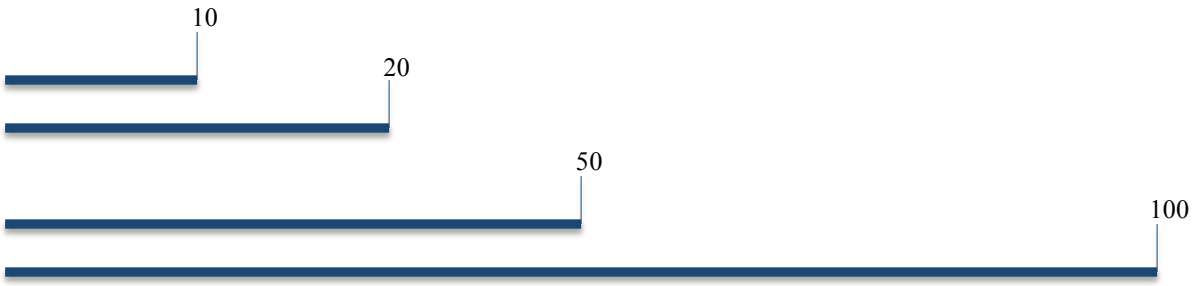
In the measurement model, two fractions are equivalent when they are located on the same point on the measurement scale.

For example, $\frac{1}{2}$ u = $\frac{2}{4}$ u as shown below. The fractions are not identical, because one is composed of 1 copy of a 2-split of the unit and the other 2 copies of a 4-split of the unit. But the resulting distance traveled from the origin is identical and the lengths of each are congruent.



Other senses of equivalence are based on relations between the units. For example, 1 inch is approximately 2 $\frac{1}{2}$ cm., so that 12 inches = 30 cm.

Later, we will explore division as re-measuring, so that, $\frac{10\ in.}{20\ in.} = \frac{50\ in.}{100\ in.}$ Note that here equivalence means the same ratios, not the same locations on a common scale of measurement.



Instruction — Fraction Arithmetic (1/4) Unit 9

Introducing 1/4
Addition
Subtraction
Multiplication
Ruler Equivalence

Introducing 1/4

Whole Group

1. Last time, we decided that we could make part of a ft. unit by folding a strip in half. We said that the result would be ½ ft. **Talk with your elbow partner and make a prediction about what would happen if found ½ of ½ ft**. How many parts would there be? What should we call one of the parts?

Individual

2. **Let's all try it out.** First make a ½ ft. (monitor children to be sure that all can do this). Then find ½ of the ½ ft (monitor children). Unfold the strip and count the number of equal parts (4). Now fold the strip back up and hold that part up.

3. **In your journal, write how we should symbolize (represent) this part as a fraction ($^1/_4$).** Remember, the bottom number tells us something and the top number tells us something different.

Whole Group

4. **Demonstrate that we write the resulting part of the foot unit as $^1/_4$ ft.** Label the bottom number as the number of parts (the splitter). Label the top number as the number of copies.

5. **Demonstrate composing the 2 2-splits**, saying and writing: I am going to take ½ of 1 ft. Match the ends and crease the midpoint, so that when unfolded the 2 congruent sections are clearly visible. Tape 1 ft. unit to the board and directly underneath it and aligned at the endpoint, the ½ unit. Then take another ½ unit and say: Now let's find ½ of ½ ft. The result is ¼ ft. Place the ¼ ft. unit directly underneath the ½ ft. unit, so that children can see visually the relationships among the units.

6. **Talk with your elbow partner: 1 ft. is ____ times as long as $^1/_4$ ft. How could you demonstrate that you are right?**

Instruction

Say: To make certain that we really have split the unit into ¼ ft., we should be able to fit 4 copies of the ¼ ft. into the 1 ft. unit. Let's see if we can do that: Iterate another copy of the ¼ ft. unit 4 times, leaving a mark after each translation, so that children can see that 4 iterations of ¼ ft. are congruent with 1 ft.

7. **Conclude**: The 1 ft. unit is 4 times as long as the ¼ ft. unit. (Write this sentence on the board. Have students copy it in their math journals and make a drawing that shows the iteration. This is important for promoting representation.) This means that if we copy ¼ 4 times we get the same length as the unit.

Individual

8. **Have every child create a ¼ unit by splitting a foot unit into equal partitions.** Make sure that every child iterates the ¼ ft. unit 4 times to establish congruency between 4 iterations of ¼ ft. and the length of 1 ft. If children are having difficulty iterating, let them generate additional copies of ¼ ft and use them to tile the 1 ft. length.

9. **Take two paper unit ft. strips and fold each into ¼ ft. Unfold the units and tape them end-to-end with no spaces.** (Check to ensure that each child can do this, assist where needed or suggest other children to assist.)

    a. Now, put your fingers like this at the very end of the strips (gesture grasping with thumb and finger. What do we call the beginning of the first strip—how far have we traveled on the strip? (0)

    b. Close your eyes. Now travel ¼ ft. by sliding your fingers. Stop when you have traveled ¼ ft. Open your eyes and see if you stopped at the right place. Now continue to travel another ¼ ft. Stop when you have traveled another ¼ ft. How far have we traveled all together? ($^2/_4$ ft.) Open your eyes and look here (Write $^2/_4$ on the board.) What did we say the bottom number told us? (The number of equal parts that we split the unit into.) What did we say the top number told us? (The number of copies.) So, let's look at the first strip. Point to one of the equal parts. How many copies of these equal parts do we have to travel if we want to go from the beginning to the middle of the first strip? (2). So that is the meaning of $^2/_4$, and because we have now traveled halfway to

the end of the first unit, we say that $^2/_4$ ft. and ½ ft. mean the same thing. They are equal: $^2/_4$ ft. = ½ ft.

c. Go back to the beginning. Close your eyes and travel ¾ ft. Now travel another ¼ ft. Where are you? What can we call where we are? ($^4/_4$ ft., $^2/_2$ ft., 1 ft. Go back to 0. Close your eyes and travel $^5/_4$ ft. Open your eyes and check to see if you are there.

d. Go back to the beginning each time, and travel $^3/_4$ ft., $^4/_4$ ft. and $^6/_4$ ft. and $^9/_4$ ft.

Whole Group

10. **The teacher leads a conversation about traveling different amounts of units**, reminding students that, for example, $^3/_4$ ft. means traveling 3 of the ¼ ft. unit and that $^4/_4$ ft. means traveling 4 of the ¼ ft. unit and that ¼ ft. means traveling from 0 to ¼ ft (Traveling only 1 copy of the ¼ ft. unit).

11. **Draw a length on the board, naming it ¼ unit**. If this (gesture to length) is ¼ unit, how long should 1 unit be? How could we tell for sure?

12. **Draw a length on the board, naming it $^2/_4$ unit**. If this (gesture to length) is $^2/_4$ unit, how long should 1 unit be? How could we tell for sure?

    Teacher note. This may be more difficult for children who approach the problem as involving first knowing that ¼ means that the length is 4 times as long as 1 (4 iterations), and second, that $^2/_4$ means that the length is twice as long (relating iterations of the composite unit $^2/_4$ to 1), so that the unit is twice as long. Other children may know that $^2/_4$ = ½ so they don't need to take this extra step, and this problem is then just like one they have solved before.

Instruction

Introducing 1/4
Addition
Subtraction
Multiplication
Ruler Equivalence

Addition

Whole Group

1. What do we do when we add two lengths? (Join them.) Talk with your elbow partner and decide what ¼ + ¼ means.

   Q: _____ come on up and show us with these strips (have ¼ ft. strips available) what ¼ ft.+ ¼ ft. means. What is the result? ($^{2}/_{4}$ ft.)

   Q: How many different ways could I add to get $^{4}/_{4}$ ft? (e.g., $^{2}/_{4}$ ft. + $^{2}/_{4}$ ft., $^{0}/_{4}$ ft. + $^{4}/_{4}$ ft., ¾ ft. + ¼ ft.)

Individual

2. Use your ft. strips to find:

   a. ¼ ft. + ¼ ft. + ¼ ft. =

   b. $^{5}/_{4}$ ft. + $^{3}/_{4}$ ft.=

   c. ¾ ft. + $^{2}/_{4}$ ft. =

Whole Group

3. Students share solution strategies and teachers support the movement between symbolization and concrete enactment of solutions. (Never just act. Never just symbolize. Always relate the symbolization to the action of joining strips and the action of joining to the symbolization.)

Instruction

Fraction Arithmetic (1/4) Unit 9

Introducing 1/4
Addition
Subtraction
Multiplication
Ruler Equivalence

Subtraction

Whole Group

1. What do we do when we subtract two lengths? (Find the difference between them) Talk with your elbow partner and decide what $^5/_4 - {}^2/_4$ means.

   Q: _____ come on up and show us with these strips (have 1 ft. and ¼ ft. strips ready) what $^5/_4$ ft. $- {}^2/_4$ ft. means. What is the result? ($^5/_4$ ft. $- {}^2/_4$ ft. = ¾ ft.)

Individual

2. Use your ft. strips to find the result in (a-c). For each, write a number sentence that would undo the result of the subtraction:

   a. $^3/_4$ ft. – ¼ ft.

   b. $^4/_4$ ft. $- {}^3/_4$ ft.

   c. $^8/_4$ ft. $- {}^4/_4$ ft.

Whole Group

3. Students share solution strategies and teachers support the movement between symbolization and concrete enactment of solutions. (Never just act. Never just symbolize. Always relate the symbolization to the action of finding the difference between lengths and for undoing the subtraction, the action of joining and symbolizing this joining as addition.

Instruction

Multiplication

Introducing 1/4
Addition
Subtraction
Multiplication
Ruler Equivalence

Whole Group

1. What is 3 of 2 ft.? How long is that? Who can show that? (Look for literal enactment of 3 of a composite unit of 2 ft.) What about 3 of 1 ft. How long is that? Who can show that? How about 3 of ¼ ft.? Who can show that? (Look for or demonstrate 3 copies of ¼ ft, which is the same as $^3/_4$ ft.)

    a. Talk with your elbow partner. What does ¾ of 1 ft. mean? The bottom number is the splitter and the top number is the number of copies. What about $^4/_4$ of 1 ft.?

    b. The teacher elicits student responses and demonstrates the meaning of the operation by splitting 1 ft. unit into 4 congruent parts, and then iterating the fractional unit length 3 times. And then 4 times. Because we can fit ¼ ft. into $^3/_4$ ft. 3 times, we can say that ¼ ft. is 3 times as long as $^3/_4$ ft.

Individual

2. Use a ft. unit. Write $^2/_4$ x 1 ft. and show how to find the result. (Teacher roves and checks student understanding.)

Instruction

Ruler Equivalence

Introducing 1/4
Addition
Subtraction
Multiplication
Ruler Equivalence

Individual

1. Use your ¼ unit to mark your ruler to show the ¼ split for every unit. Label your ruler so that someone else could easily see where $^{3}/_{4}$ ft. or $^{5}/_{4}$ ft. is.

2. Use your ruler to measure the snakes. How long are those snakes? (The snakes have lengths of $^{14}/_{4}$ ft. and $^{10}/_{4}$ ft.)

Whole Group

3. What labels on your ruler were different numbers but meant the same thing—they meant the same exact length from the starting point? ($^{2}/_{4}$ ft.= ½ ft.; $^{4}/_{4}$ ft.= $^{2}/_{2}$ ft. = 1 ft. etc.)

   Teacher note. The intention is to start to build some landmarks, such as $^{2}/_{4}$ = ½, 1 = $^{2}/_{2}$ or $^{4}/_{4}$. We intend to support a generalization of n/n = 1.

Formative Assessment

Fraction Arithmetic (1/4) Unit 9

Name _____

1. For each problem, write the result and show how you found it with the length units provided. You can tape the units on the paper.

(a) $\frac{1}{4}$ of 1 unit

(b) $\frac{1}{4}$ u + $\frac{1}{4}$ u + $\frac{1}{4}$ u

Formative Assessment

Fraction Arithmetic (1/4) Unit 9

(c) $\frac{3}{4}$ u **+** $\frac{4}{4}$ u

(d) $\frac{5}{4}$ **-** $\frac{2}{4}$

Formative Assessment

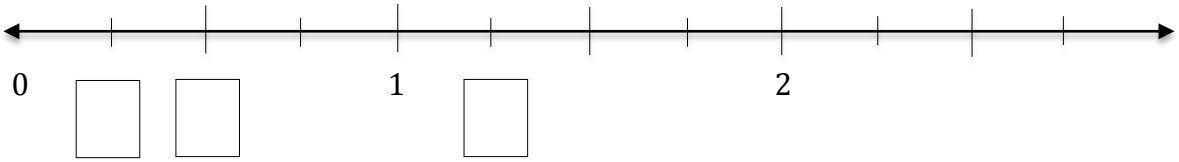
(e) $\frac{5}{4}$ **of 1 unit**

2. Put these into order from shortest length to longest length:

$\frac{5}{4}$ ft. $\frac{2}{4}$ ft. $\frac{1}{2}$ ft. $\frac{3}{4}$ ft. $\frac{1}{4}$ ft.

Formative Assessment

3. What numbers go in the boxes on this numberline?



4. For each, decide if the expression is true or false.

(a) $\frac{2}{4}\,\mathbf{u} = \frac{1}{2}\,\mathbf{u}$

(b) $\frac{4}{4}\,\mathbf{u} = \frac{2}{2}\,\mathbf{u}$

(c) $\frac{6}{4}\,\mathbf{u} = \frac{4}{2}\,\mathbf{u}$

Formative Assessment Record — Fraction Arithmetic (1/4) Unit 9

Student _____ Date _____

Indicate the levels of mastery demonstrated by circling those for which there is clear evidence:

Construct/Concept	Evidence for	Notes
ToML 4E **¼ of 1**	Student folds strip unit into 4 congruent parts. Names 1 of the congruent parts as ¼.	
Addition as Joining of Lengths Fractions>1 **¼ + ¼ + ¼** **¾ + $^4/_4$**	Student joins 3 4-split units, says result is $^3/_4$. Student joins 7 4-split units, says result is $^7/_4$.	
Multiplication of a Measured Length by a Fraction: N copies of 2-split **$^5/_4$ of 1 unit**	Student shows 5 iterations of ¼.	
Subtraction as Difference $^5/_4 - {}^2/_4$	Student shows difference between 5 ¼ unit strips or 1 ¼ units and $^2/_4$ unit strip is ¾ units.	
Order fractional measures	Student orders fractions from least to greatest.	
Number line Representation	Student locates ½ and ¼ correctly and labels the 1 ¼ either as $^5/_4$ as 1 ¼.	
Equivalence	Identifies ½ unit and $^2/_4$ unit as equivalent Identifies $^4/_4$ unit and $^2/_2$ units as equivalent Identifies $^6/_4$ unit and $^4/_2$ unit as equivalent	

Academic Language: Indicate academic words the student is familiar with by recording them here.

