Justifying and Using Formulas to Find Area Measure of Triangle, Regular Polygon, and Circle

UNIT

5

Mathematical Concepts

- The measure of the area of a triangle, $\frac{1}{2}b \times h$, is $\frac{1}{2}$ the product of the length of a side and its corresponding height. The height is the shortest distance between the opposite vertex and the side chosen as the base, or the line containing the side chosen as the base if the opposite vertex is not over the base.
- The measure of the area of a regular polygon, $\frac{1}{2}\ P\times\ a$, is $\frac{1}{2}$ the product of its perimeter, P, and its apothem, a. The apothem is the shortest distance from the center of a regular polygon to a side.
- The measure of the area of a circle, πr^2, is the product of π and the square of the circle's radius.

Unit Overview

In light of previous consideration of the area of a parallelogram as $b \times h$, the area of a triangle partitioning a parallelogram into 2 congruent triangles is established as $\frac{1}{2}$ the area of the parallelogram, or $\frac{1}{2}b \times h$. Since any triangle can be $\frac{1}{2}$ turn-rotated about one of its sides and composed to form a parallelogram, the formula is general. Working from this foundation, the area of a regular hexagon is considered by partitioning it into six congruent triangles. By summing the area of each of these six congruent triangles, the area of the hexagon is established as $\frac{1}{2}P \times a$, where P represents the perimeter of the hexagon and a the height of any of the six congruent triangles. As the number of sides of a regular polygon increases without limit, the polygon increasingly resembles a circle. Considering a circle as a regular n-gon with an infinite number of sides, the perimeter (the circumference of the circle) is π x d and its apothem is the radius of the circle. This results in the familiar formula for the area of a circle, π x r^2, because by analogy to the formula for the area of a regular polygon, the area of the circle is $\frac{1}{2}$ x π x $(2r)\times r$.

Contents

Materials and Preparation

Read

- **Unit 5**
  Start by reading the unit to learn the content and become familiar with the activities.

Gather

- Rulers with inch markings or with centimeter markings
- Patty paper

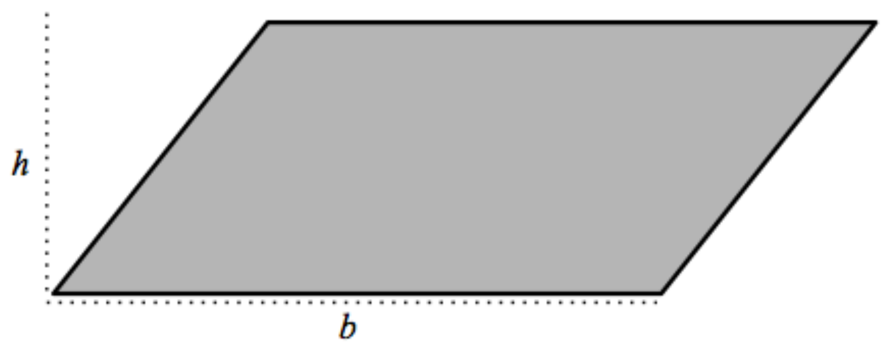
Academic Vocabulary

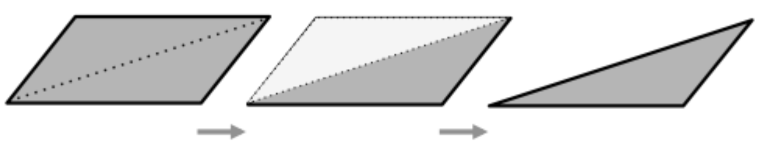
- Area
- Length
- Width
- Rectangle
- Perimeter
- Regular Polygon
- Pi
- Unit
- Greater than
- Less than
- Equal to
- Partition
- Apothem
- Radius

Mathematical Background

Area of a Triangle

Recall that in Unit 4, we established the measure of the area of a parallelogram as the product, *base* × *height*, which was justified by considering the area of a rectangle as $b \times h$, and by Cavalieri's Principle, the area of a parallelogram must also be $b \times h$. Because a parallelogram can be formed by composing a triangle and its $\frac{1}{2}$-turn rotated image, the area of a triangle must be $\frac{1}{2} b \times h$, as shown below. Recall h is the distance perpendicular to the selected base from the opposite vertex.





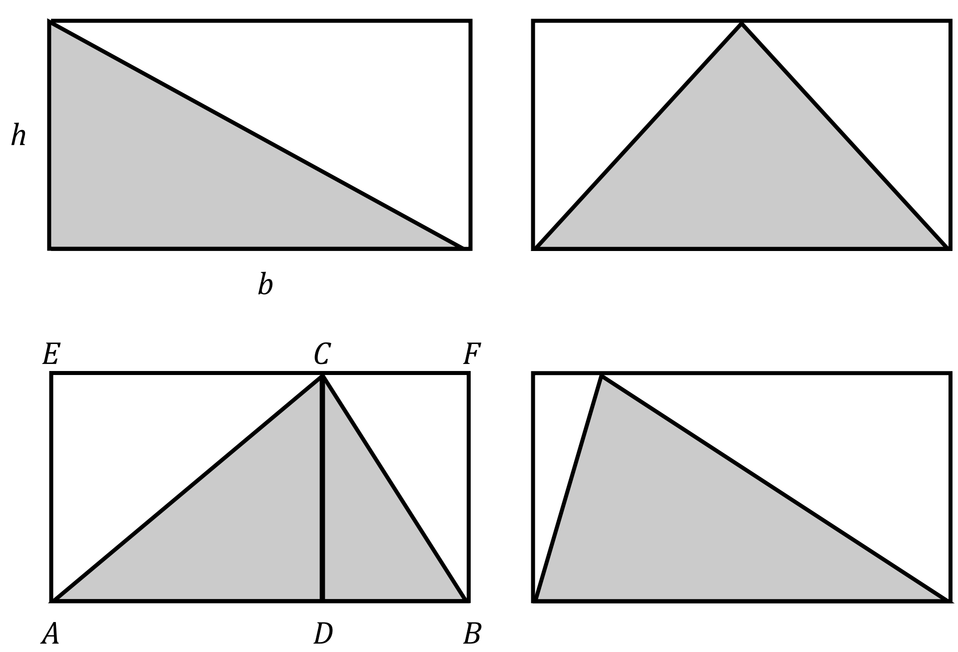
Mathematical Background

Another justification for the measure of the area of a triangle can be generated by considering the area measure of a rectangle, $b \times h$. Consider a rectangle as composed of two right triangles, as depicted by the upper left panel in the figure below. Selecting the midpoint of the hypotenuse of the shaded triangle and rotating about that point $\frac{1}{2}$ turn demonstrates that the triangles are congruent. This suggests that the area of a triangle is $\frac{1}{2}\ base \times height$. Consider other potential triangles, such as those displayed in the remaining panels of the figure.

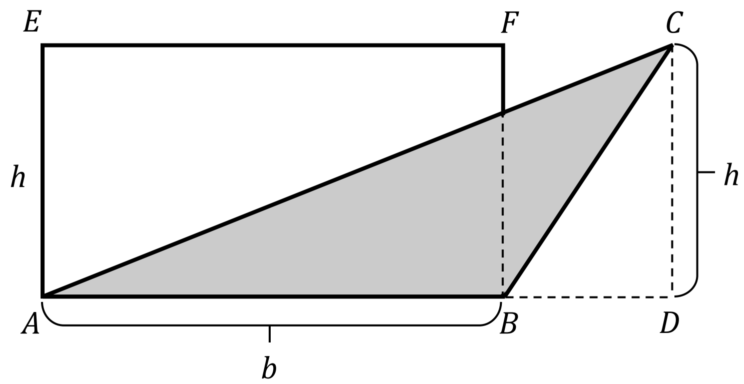
For example, in the lower left panel:

- the area of Δ ABC $=$ area Δ ADC $+$ area Δ CDB;
- the area of Δ ADC $= \frac{1}{2}$ area AECD;
- and the area of Δ CDB $= \frac{1}{2}$ area DCFB;

hence, the area of Δ ABC $= \frac{1}{2}$area AEFB $= \frac{1}{2} b \times h$.



For further conviction, consider cases of triangles not enclosed in the rectangle.

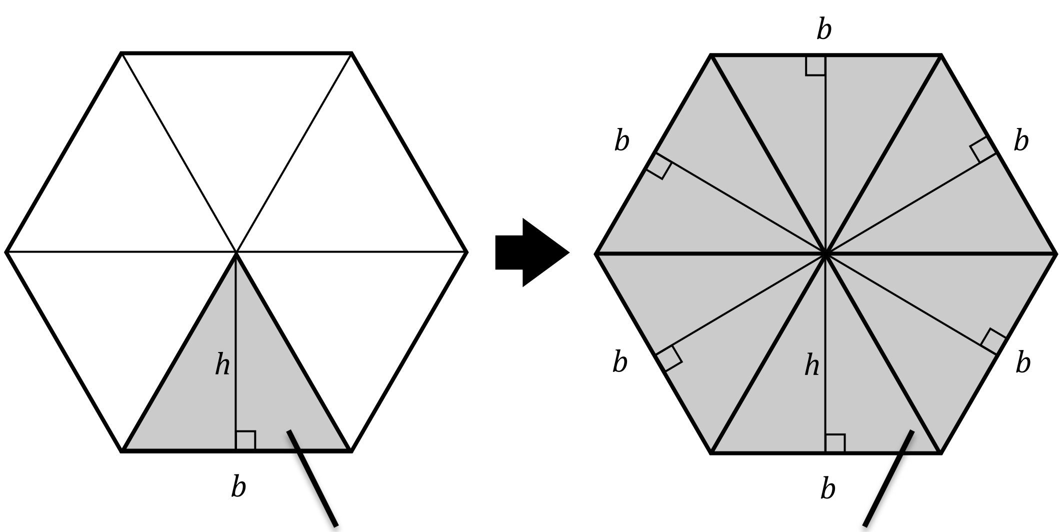


Area of Δ ABC = Area (Δ ADC) − Area (Δ BCD)

This difference is $\frac{1}{2}$ area (AECD) − $\frac{1}{2}$ area (DCFB), which is the same as $\frac{1}{2}$ area (AEFB), and therefore $\Delta \text{ ABC} = \frac{1}{2}\ b \times h$.

Area of a Regular Hexagon

Given the measure of the area of a triangle, a strategy for finding the area of any polygon is to dissect it into triangles. The area measure of the polygon is then the sum of the measures of the areas of its constituent triangles. Because the triangles for regular polygons are congruent, the area of a regular polygon can be found as illustrated below for a regular hexagon. The altitude or height of the equilateral triangle is also called its apothem.



Area of Triangle $= \frac{1}{2} \times b \times h$

Area of Hexagon $= 6 \times \textit{Area of Triangle}$
$= 6 \times (\frac{1}{2} \times b \times h)$
$= (6 \times b) \times \frac{1}{2} \times h$
$= P \times \frac{1}{2} \times h$
$= \frac{1}{2} P \times h$

Area of a Circle

Consider that as the number of sides of a regular polygon grows, the polygon approximates a circle more and more closely. As the number of sides grows without limit, the perimeter of the polygon and the circumference of the circle become indistinguishable. This means that we can use our knowledge of the relation between the circumference and the diameter of a circle to obtain a measure of the circumference. The measure of the length of the circumference, *C*, of a circle (its perimeter) is πd, where π is about 3.14 and *d* is the diameter of the circle. The apothem of the *n*-gon and radius, *r*, of the circle are now also indistinguishable, so the area of the circle is $\frac{1}{2}\ C \times r$. But $C = \pi$ x $(2r)$, leading to $\frac{1}{2}$ x $\pi\ x\ (2r) \times r$, so the area of the circle is π x r^2, often abbreviated as πr^2.

Individual

Each student receives a copy of the parallelogram worksheet, a piece of patty paper and a ruler.

Find the area of the parallelogram.

Whole Group

Share and compare solution strategies.

Teachers note: Be sure to elicit justifications for strategies used, especially area measure as $h \times b$, where h is height and b is the length of the base.

Individual

Partition the parallelogram along one of its diagonals into two congruent triangles. Use patty paper to trace the outline of one triangle. Then use the midpoint of the diagonal to $\frac{1}{2}$ turn the triangle. What do you notice? If the area of the parallelogram is $b \times h$, what must the area of the triangle be?

Draw a triangle, however you like. Trace it with the patty paper. Then $\frac{1}{2}$ turn it about one of its sides to construct a parallelogram. Make a conclusion about the area of a triangle and its measure.

Whole Group

Share and compare the results of the investigation of triangles and parallelograms.

Teacher note. Help students understand that if we accept the measure of area of a parallelogram, then by using a transformation ($\frac{1}{2}$ turn), for any triangle we can establish its area measure. Recall that all parallelograms have $\frac{1}{2}$ turn symmetry, and that all parallelograms can be dissected into 2 triangles. An alternative pathway for arriving at the same conclusion can be established by starting with a rectangle, as shown in Mathematical Background.

Individual

Find the areas of the triangles depicted in the Worksheet, Area of a Triangle.

Teacher note. You could assign each student a different side of each triangle and then rely on the Whole Group discussion to highlight how to find the height of each corresponding side.

Whole Group

Share and compare solutions to the problem of identifying the height associated with each side.

Teacher note. Emphasize how to find the height of a triangle from any side of a triangle. Challenge students to come up with a method for finding the height (by extending the side) when the opposite angle is not over the side. Within measurement error, the area measures should be identical for each triangle, no matter which side is selected as its base.

Partner

Construct a regular hexagon using a protractor and a ruler, or construct a triangle (what kinds of triangles work?) and use one or more transformations of the triangle to construct the hexagon.

Teacher note. You might split the class so that some portion attempts to first construct a triangle and then use patty paper to transform it to form a regular hexagon, while the remainder of the class constructs the hexagon as a path. If the starting point is a triangle, the sum of the interior vertex angles at the center of the hexagon composed of 6 congruent triangles must be 360 degrees. (See the geometry lessons on tiling). Only equilateral triangles will meet this criterion, although this will likely be more visible if other types of triangles are attempted. A path perspective on the construction of a regular hexagon consists of 6 moves of the same length and 6 turns, each of 60 degrees, because the total turn is 360 degrees, and hence $360 \div 6 = 60$. Similarly, an equilateral triangle is constructed as a path consisting of 3 congruent lengths and turn angles of 120 degrees.

Whole Group

Compare constructions with an eye toward establishing how the method results in a regular hexagon.

Partner

Invent a strategy for finding the area measure of the regular hexagon.

Whole Group

The teacher selects student strategies that use some form of dissection, such as partitioning into 2 trapezoids or 6 triangles. Other strategies that may not work but which will lead to productive discussions are also highlighted.

Partner

Partners use a dissection to find the area measure of the hexagon.

Whole Group

Share and compare solution strategies.

Teacher note: Be sure to include strategies that make use of congruent triangles. When students indicate that they understand this approach, have them try it out on another hexagon. This is a great opportunity to revisit the path perspective of shape by having students use a ruler and a protractor to construct a regular hexagon. (Recall that a regular hexagon is composed of 6 congruent lengths and 6 turn angles of 60 degrees. Then try to help students develop a general formula for the area of a regular hexagon: $\frac{1}{2}\ P \times apothem$ (altitude or height of each triangle).

This relies on: $\frac{1}{2}$ x h x b_1 $+\frac{1}{2}$ x $h\ x\ b_2$ $+\frac{1}{2}$ x h x b_3 $+\frac{1}{2}$ x h x b_4 $+$ $\frac{1}{2}$ x h x b_5 $+\frac{1}{2}$ x h x b_6

Note that each b is multiplied by $\frac{1}{2}$ x h. By the distributive property of multiplication over addition, we can express this as:

$\frac{1}{2}$ x $h \times (b_1\ +\ b_2\ +\ b_3\ +\ b_4\ +\ b_5\ +\ b_6)$

An equivalent expression is then $Area = \frac{1}{2}$ x $h \times P$, where the perimeter, P, is the sum of the lengths of the bases of the hexagon.

Partner

What is the relationship between the length of the circumference of a circle and its diameter?

Directions to students: **Construct three different circles** with a pencil and string on paper, or use a compass, or trace the outline of a cylindrical container for this construction. For each circle, use your ruler to find the measure of its diameter and the measure of its circumference (use rope caulk to trace the outline of the circumference and then unfold it and use your ruler to find its length, or use a long paper strip to trace the circumference). Record values of diameter and circumference for each circle.

Whole Group

Create a table of values of diameter, circumference, and how many diameter lengths are in each circumference (C/D). Find the median or mean value of C/D. It should be close to 3.14.

Teacher note: Demonstrate $C = \pi \times D$ for a prototypical circle by first cutting a length of rope caulk, or a length of string, or a paper strip, that is congruent with the diameter. Then use it to predict how long the circumference of the circle will be, iterating the rope caulk or string-length or paper strip, that many times. If students are used to implicit multiplication by juxtaposing letters, then C = πD.

Partner

Predict and test. Construct a new circle with a diameter of your choice. Using the diameter, predict the length of the circumference.

Then find the area of the circle as $\frac{1}{2}$ x $P \times a$, where $P = \pi \times 2r$ and $a = r$.

Whole Group

Students share solution strategies.

Teacher note: Help students arrive at Area = $\pi \times r \times r = \pi r^2$. Help students understand the formula as a generalization of the formula for a regular polygon. Recall that the area of a regular n-gon is $\frac{1}{2} aP$ and now $a = r$ and P = $\pi 2r$, so $\frac{1}{2}$ $r\pi$2r = $\frac{1}{2}$ $2rr\pi = \pi r^2$.

Formative Assessment

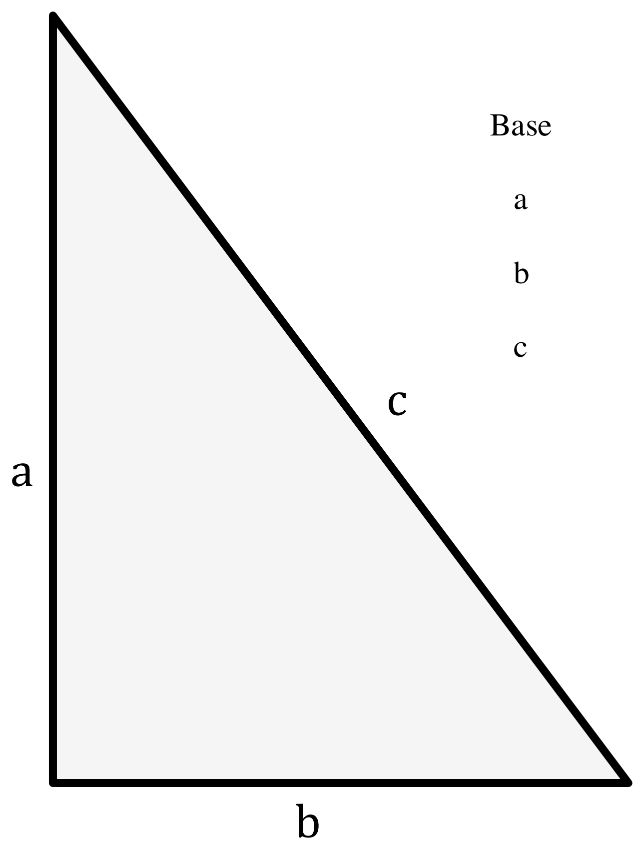
Administer the formative assessment and select contrasting student responses to create further opportunities for learning about area measure, especially the difference between units of length measure (perimeter) and units of area measure.

Formative Assessment

Name:____

1. A triangle has three sides. Explain why the formula for the area measure of a triangle is $\frac{1}{2}\ b \times h$ and not $\frac{1}{3}\ b \times h$. Your explanation can include a drawing.

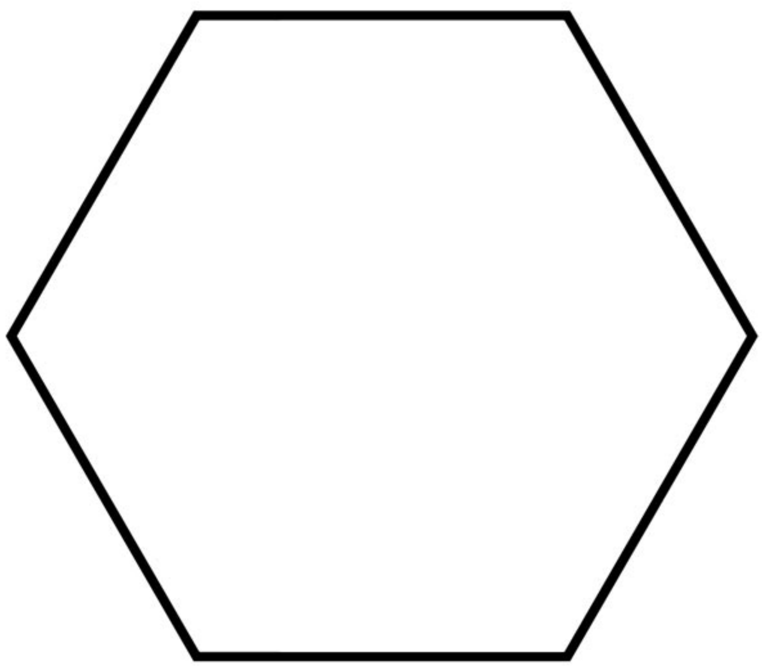
2. In the triangle below, find the heights for each side in the formula $A = \frac{1}{2}\ b \times h$. Draw the height when side c is the base.



Base	Height
a	_____
b	_____
c	_____

What is the area of this triangle?

3. The measure of the area of a regular hexagon is $\frac{1}{2}\,P \times a$, where P is the perimeter of the hexagon and a is its apothem (the shortest distance from the center of the hexagon to the opposite side). Find the area measure of this regular hexagon. Show your work.



4. Explain why the area measure of a regular hexagon is $\frac{1}{2}\,P \times a$, where P is the perimeter of the hexagon and a is its apothem.

5. The length of the circumference of a circle is ______ times as along as its diameter.

6. If the radius of a circle is 5 cm., what is its circumference? It's area?

7. Explain how the thinking about the formula for the area of a regular polygon, $A = \frac{1}{2} P \times a$, justifies the formula for the area of a circle, $A = \pi r^2$.

Formative Assessment Record — Area Unit 5

Item	Level Circle highest level of performance	Description Circle each criterion of performance met by student	Notes
Item 1 Explain/justify formula for area of a triangle.	**ToAM 5B** Generate, use, and explain area formula for a triangle.	Justifies by appeal to area of a parallelogram or rectangle as b x h and shows or says that triangle formed by splitting along the diagonal is $\frac{1}{2}\ b \times h$.	
	NL	Cannot justify.	
Item 2 Identify potential bases and associated heights of a right triangle.	**ToAM 5B** Generate, use, and explain area formula for a triangle.	Identifies height of base a as side length b. Identifies height of base b as side a. Draws height of base c as a perpendicular line segment through opposite vertex (the vertex formed by the intersection of sides a, b) Uses formula to find area.	
	Other Describe		
Item 3 Find area of regular hexagon	**ToAM 5C** Generate, use, and explain area formula for other polygons (e.g., hexagon).	Finds perimeter of hexagon. Finds apothem of hexagon. Area as $\frac{1}{2}\ P \times a$. Divides hexagon into 6 eq. triangles and finds area of one, finds area as 6 × that area.	
Item 4 Explain formula for area of a regular hexagon.	**ToAM 5C** Generate, use, and explain area formula for other polygons (e.g., hexagon).	Dissects hexagon into 6 eq. triangles Area of a triangle as $\frac{1}{2}\ b \times h$. Sum of bases is Perimeter, so $A\ = \frac{1}{2}\ P \times h$ (or a) [or equivalent expression]	
Item 5 C = π x D	**ToAM 5D** Generate, use, and explain area formula for circle (characterized as *n*-gon).	The length of the circumference of a circle is (3 or 3 1/10 or other reasonable approximation) times as long as its diameter.	
Item 6 Find circumference, area of circle with radius 5 cm.	**ToAM 5D** Generate, use, and explain area formula for circle (characterized as *n*-gon).	Understands that C = 2*r*. Finds area as 25 π or reasonable approximation.	
Item 7 Explain formula for area measure of a circle.	**ToAM 5D** Generate, use, and explain area formula for circle (characterized as *n*-gon).	Suggests perimeter of an n-gon and circumference are equivalent if n sides keeps increasing (or come closer and closer together). Suggests radius and apothem (or height of eq triangle in an *n*-gon) are equivalent. Establishes that since C = 2*r*π, then $\frac{1}{2}\ C \times r\ =\ \pi r \times r$ or πr^2	

Worksheet Area of a Parallelogram

NAME:____

Find the area of this parallelogram:

Area = _________

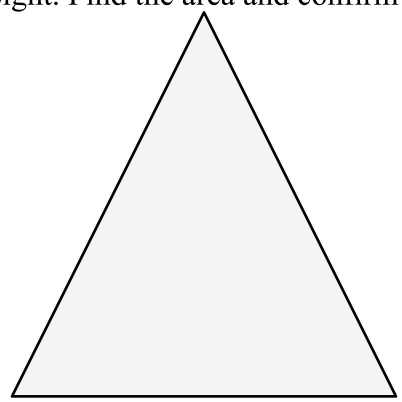
Worksheet Area of a Triangle — Area Unit 5

Mathematical Concepts
Unit Overview
Materials and Preparation
Mathematical Background
Instruction
Formative Assessment
Formative Assessment Record
Worksheets
Area of a Parallelogram
Area of a Triangle

NAME:____

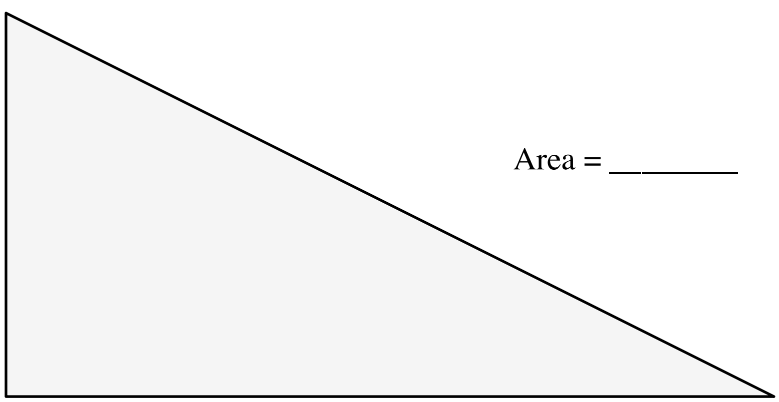
Find the area of each triangle by choosing one side as the base, and finding its height. Then choose a different side as the base and find its height. Find the area and confirm that they are the same measure.

1.



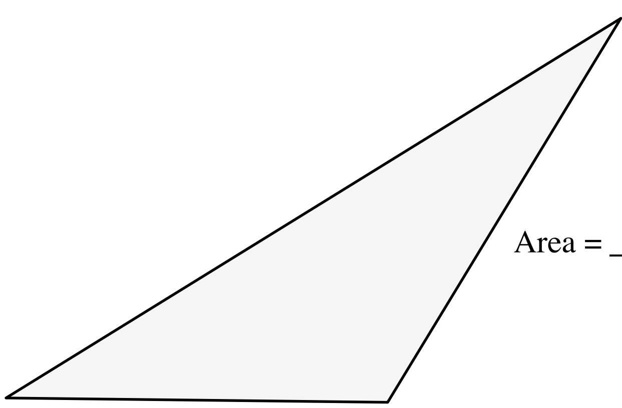
Area = ________

2.



Area = ________

3.



Area = ________