Partitioning and Comparing Rectangles

UNIT

1

Contents

Mathematical Concepts

- We call the space enclosed by a 2-dimensional figure an area.
- A 2-dimensional figure A can be partitioned (dissected) into two or more pieces. If the pieces of figure A cover all of figure B without any leftovers or overlaps, we say that the areas of the figures are congruent.
- A subdivision can be privileged to constitute a unit of measure. For example, a square or rectangular or triangular partition can serve as a unit of area measure.
- The measure of the area of a figure is the ratio of the area of the figure to the area of a unit. Practically, this is established by counting the number of units that cover the figure.
- Areas of different figures can be compared without re-arranging pieces by counting units, if the units are identical and tile the plane (as in units consisting of squares, rectangles, or triangles).

Unit Overview

This unit encourages students to spatially structure, and re-structure, 2-dimensional spaces as they compare the space covered (area) of three different-looking rectangles. Without using rulers or other metrics, students partition the rectangles and attempt to establish relations among the rectangles by re-arrangement of the partitions. Students typically propose privileging one of the partitions, most often a rectangle, but occasionally a square, and use the count of that unit to compare the space covered by the three rectangles. The unit ends with student investigation of re-arrangements of 12 unit squares to produce shapes with the same area but often with different perimeters and of different appearances. The formative assessment is aimed at firmly differentiating units of length measure (perimeter) from units of area measure.

Materials and Preparation Area Unit 1

Read

- **Unit 1**
  Start by reading the unit to learn the content and become familiar with the activities.

Gather

- Sets of three rectangles constructed from unmarked chart paper with (horizontal x vertical) dimensions of: 12×1, 4×3, and 2×6. A template for each rectangle is found in the Appendix. Label the rectangles A, B, and C (See Figure 1). Rectangles must have no folds or tape lines. Do not laminate or use paper with lines, grids, or other markings.
- If you are working with a group or a whole class, make larger rectangles with the dimensions scaled as:
  - A = 48 *in*. × 4 *in*.
  - B = 16 *in*. × 12 *in*.
  - C = 8 *in*. × 24 *in*.

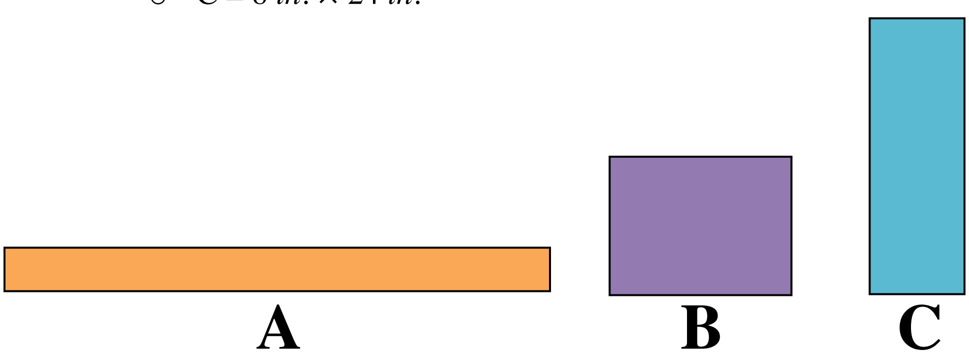


Figure 1. Rectangles A, B, and C

- Unit squares consistent with the dimensions of the smaller rectangles (1 *in*. × 1 *in*.)
- Markers
- Chart paper for drawing figures and recording dimensions
- Scotch tape or markers for affixing rectangles to the board
- Teacher math-journal (for note-taking)
- Camera (optional, for documenting student activity)
- Student math journals

Materials and Preparation Area Unit 1

Prepare

- Three large rectangles that the whole class can see and refer to.
- Three rectangles for each pair of students
- 12 identical unit squares or 6 identical unit rectangles for each pair.
- Space for pairs to work together relatively independently
- 1 T-chart per pair labeled “Area” and “Perimeter”
- Make markers available

Academic Vocabulary

- Area
- Length
- Width
- Rectangle
- Perimeter
- Congruent

Unit
Greater than
Less than
Equal to
Split
Length

Mathematical Background Area Unit 1

The equality of area of 2-dimensional figures is established by demonstrating that they are additively congruent or by counting the same number of identical area units.

Additive Congruence/Dissection

If a 2-dimensional figure can be folded and its partitions rearranged so that they completely cover a second figure, the two figures are said to be *additively congruent*, and they have the same area. Another way to say this mathematically is that they are equivalent by dissection. One can be cut up into a finite number of pieces, and these pieces can be rearranged to form the second figure.

Area Measure

Area measure is a ratio of the space enclosed by a plane figure and a unit of measure. Even though two figures may look very different, they may still have the same area, covering the same amount of space.

Unit of Area Measure

Units of area measure (typically squares or rectangles) are consistent-sized partitions of 2-dimensional space that can be counted and composed to quantify the area enclosed by a figure. Measuring all of the figures with the same unit is a way to compare their areas.

Perimeter

The linear distance all the way around a figure is its perimeter. This distance can be thought of as a path around a figure. Perimeter is measured with units of length, in contrast to area's measure with 2-dimensional units.

Whole Group

The teacher presents the three rectangles to the class and pairs of students (or each student) receive a scaled copy of the set. For whole group, the teacher attaches each rectangle to the board with magnets or tape so students can make comparisons among them. She tells students she is making a quilt or rug by cutting and sewing together remnants (pieces). Each piece costs the same amount, so she wants to buy the piece that covers the most space, so she gets the most for her money. She explains that the three pieces of paper represent the 3 sizes the material is sold in. She asks students to help figure out which piece to buy to get the most material.

Teacher Note: The teacher facilitates the discussion, asks clarifying questions about student statements, asks for student reasoning or definitions, and juxtaposes students' ideas to promote mathematical argument around the structure and measurement of the three rectangles. Students are permitted to fold the pieces but not to use rulers or other tools. The aim is to support strategies of additive congruence (dissection) — meaning that students split the area into parts and rearrange the parts to establish the relative amounts of space covered by each rectangle.

Discussion Questions

> Q: Which piece of material covers the greatest amount of space? Why do you think so?
> Q: How can we compare the space without cutting it?
> Q: If there is a proposal that they all cover the same amount of space: How can three rectangles that look so different cover the same amount of space? How could I demonstrate that they do?

Pair Work

Students work in pairs to consider which of the three rectangles covers the most space. As students work in pairs, the teacher roams and prompts student thinking:

> Q: If students develop a consistent partition to establish that the rectangles have the same measure:
> Is there another one that would work?
> Q: If students have developed a rectangle as a unit of measure and no other solution seems to use a square, ask at least one pair:
> Is there a square that might work for comparing the space enclosed by the rectangles?

Whole Group

Allow students to explore these various methods (without cutting or physically disassembling the rectangles) and have them share and compare strategies. After students have shared their strategies, ask them to consider which methods would work all of the time or with any set of rectangles. If student strategies have not included a unit (e.g. a rectangle), then introduce this idea to them and ask what its advantages might be. It is also helpful to compare how different partitions lead to different units. For example, the space covered by rectangle A can have a measure of 12 (1 *in*. x 1 *in*.) squares, or 6 (1 *in*. by 2 *in*.) rectangles or 3 (1 *in*. by 4 *in*.) rectangles. *Point out that although the amount of space covered (its area) is not changing, its measure is*.

Exploration and Extension

After students create a unit of measure and agree to privilege that unit, give everyone 12 identical units (if squares, or 6 if 1 x 2 rectangles). Ask students to find at least 5 different ways the same amount of space can be covered. For each configuration of units, students should record the perimeter and the area. Which configuration has the least perimeter? The most? Why? Record findings in math journal or on worksheet.

Formative Assessment

Administer the formative assessment and select contrasting student responses to create further opportunities for learning about area measure, especially the difference between units of length measure (perimeter) and units of area measure.

STUDENTS' WAYS OF THINKING

- Students may fail to specify attributes, simply insisting instead that one or another rectangle is "bigger." If so, ask students to clarify what they mean by "big."
- Students may focus on one dimension, such as "tallest" or "longest" to justify their choice. If so, promote the idea that although one rectangle is longer, another may be wider, and that both length and width need to be considered together.
- Students may suggest a number of different methods for comparing the area of the rectangles. They may initially try to superimpose one onto another. If so, you may ask, "But what about the part of Rectangle X that isn't covered by Rectangle Y?"
- Students most typically fold subsections of one rectangle and use congruence to compare that section to a section of another rectangle (equality of subsections. Note that the subdivisions do not need to be squares. See Typical Student Strategies (next page).
- Students may match unequal subsections to parts on each of the other rectangles until all the space is accounted for. They may need support to account for all the parts as they superimpose them on each other. See Typical Student Strategies.
- As students reconfigure the space by folding and comparing one piece to another, units may emerge. Folds may create an array of identical squares or rectangles that can be used as units to find the measure of any space. Teachers should support the action of unit creation and use it to discuss "privileging a partition." See Typical Student Strategies.

Typical Student Strategies for Partitioning and Re-arranging Equal Parts

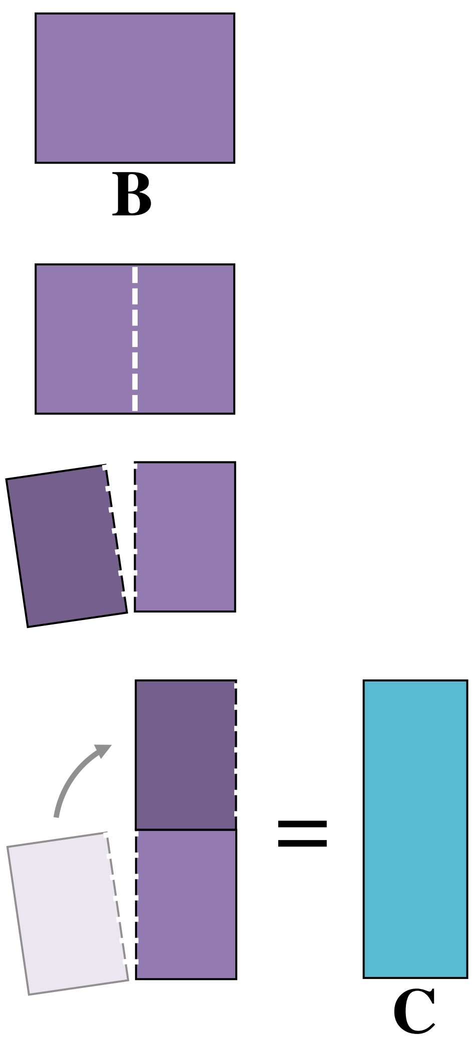
Rectangle **B** rearranged to rectangle **C**

$\frac{1}{2}\mathbf{B} = \frac{1}{2}\mathbf{C}$

$2\ (\frac{1}{2}\mathbf{B}) = 2\ (\frac{1}{2}\mathbf{C})$

$\mathbf{B} = \mathbf{C}$

B = 4×3, so the half split is 2×3. When $\frac{1}{2}$ B (or 2×3) is slid up and over the other half of B, the space is rearranged to rectangle C, 2×6.



Additive Congruence

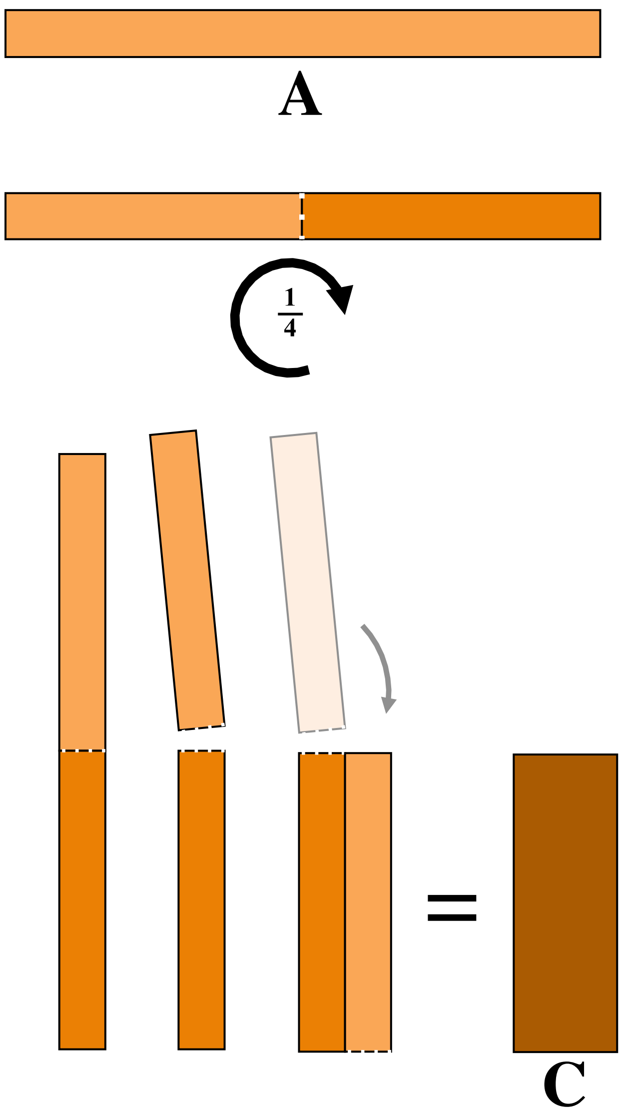
Rectangle **A** rearranged to rectangle **C**

$\frac{1}{2}$ **A** = $\frac{1}{2}$ **C**

2 ($\frac{1}{2}$ **A**) = 2 ($\frac{1}{2}$ **C**)

A = **C**

A = 12×1, so the half split is 6×1. When **A** is rotated $\frac{1}{4}$ turn right and $\frac{1}{2}$ **A** is slid over and down, the space is rearranged to rectangle **C**, 2×6.



Rectangle **A** rearranged to rectangle **B**

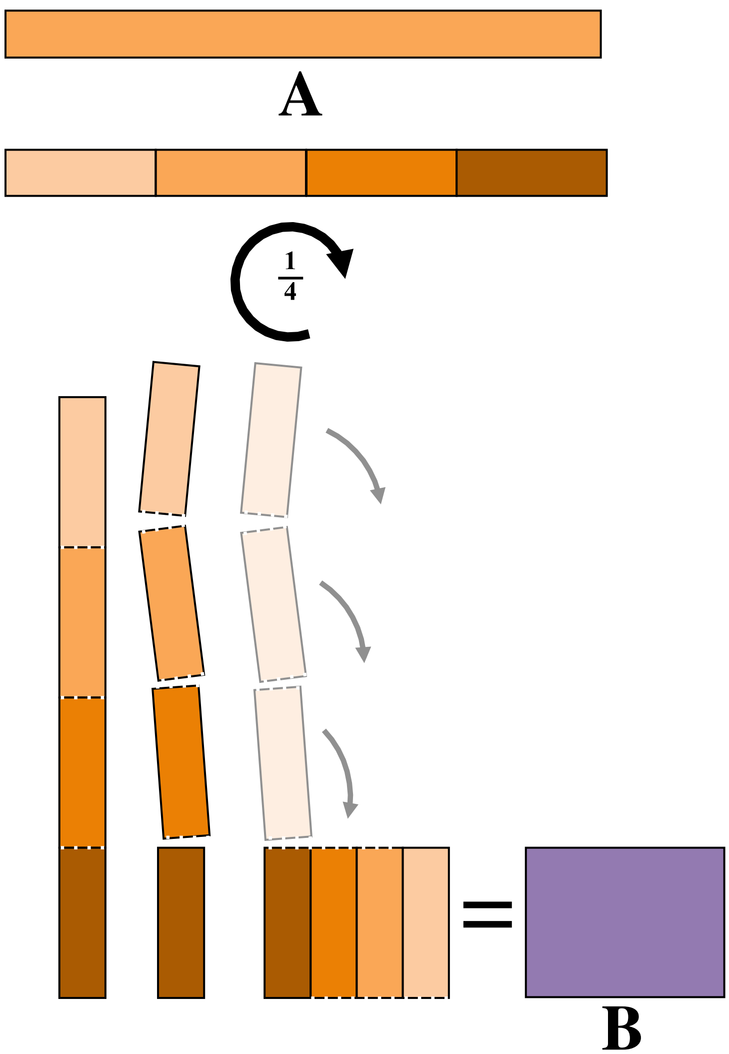
$\frac{1}{4}\mathbf{A} = \frac{1}{4}\mathbf{B}$

$4\left(\frac{1}{4}\mathbf{A}\right) = 4\left(\frac{1}{4}\mathbf{B}\right)$

$\mathbf{A} = \mathbf{B}$

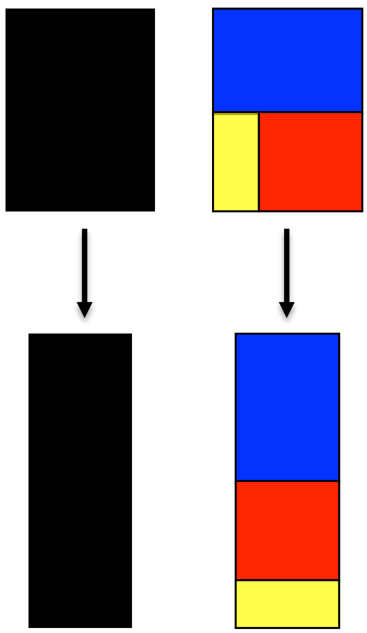
$\mathbf{A} = 12 \times 1, so \frac{1}{4}\mathbf{A} = 3 \times 1$

When A is rotated $\frac{1}{4}$ turn right, and each forth is slid so the long side is aligned with the next fourth, A is rearranged to rectangle B, 4×3.



Additive Congruence Using Unequal Parts

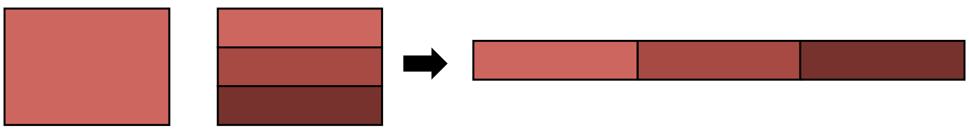
Equivalence of B (3 × 4) and C (6 × 2) established via dissection but with unequal parts.



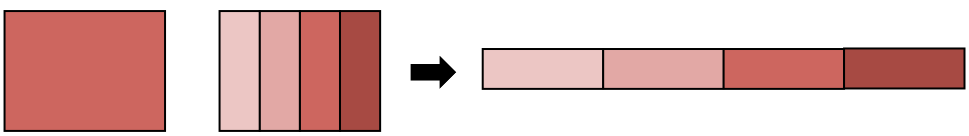
Measurement Congruence

Structuring a Square Measurement Unit from Rectangular Folds

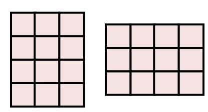
Fold thirds of **B** (12 in. x 16 in.) horizontally (each is 4 in. x 16 in.) and use the $\frac{1}{3}$ of **B** to measure, for example, **A**.



Then fold fourths of **B** vertically (each is 12 in. x 4 in.) using a composition of 2-splits to measure, for example, **A**.



Unfold rectangle **B**. It is composed of 12 squares (each is 4 in. x 4 in.). Notice that rotating **B** does not affect its area.

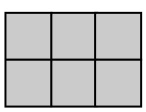


Formative Assessment Area Unit 1

NAME:______________________________

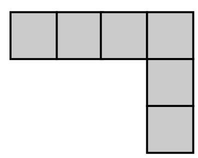
1. Use your ruler in the space below to draw an inch and a square inch.

2. Each of these shapes was made by putting together square inches (pretend that each square is 1 inch on every side). What is the area of each figure? What is its perimeter?



Area = ____in^2

Perimeter = ____in



Area = ____in^2

Perimeter = ____in

3. The area of a rectangle is measured with square inches and with square centimeters. One inch is about $2\frac{1}{2}$ times as long as one centimeter.

   Circle the statement that is true.

   a. The measure of the area of the rectangle is greater with square inches than with square centimeters.
   b. The measure of the area of the rectangle is greater with square centimeters than with square inches.
   c. The measure of the area of the rectangle is not affected by the choice of square inch or square centimeter.

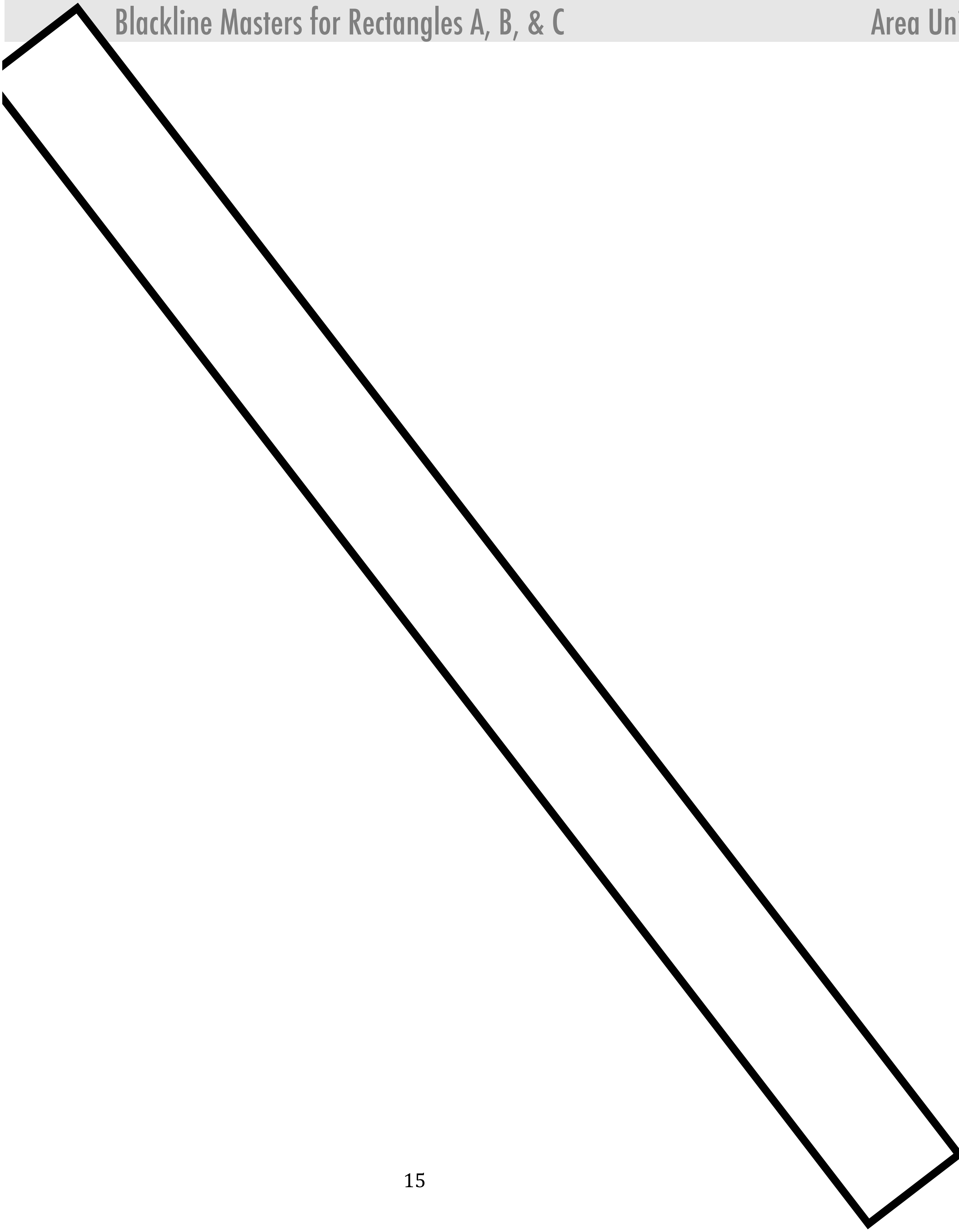
Tell why:

Formative Assessment Record — Area Unit 1

Indicate the levels of mastery demonstrated by circling those for which there is clear evidence:

Item	Level Circle highest level of performance	Description	Notes
Item 1 Constructing and Differentiating Inch and Square Inch.	**ToAM 3D** Recognize/construct suitable units.	Distinguishes between inch and square inch.	
	NL	Cannot construct the distinctions.	
Item 2 Using counts of units for area, edges of squares for perimeter.	**ToAM 3D** Recognize/construct suitable units.	Area as 6 $in.^2$ and perimeters as 10 *in*., 14 in.	
	Other Describe		
Item 3 Extension: Constructing and Differentiating fractional length and area.	**ToAM 3E** Flexible conceptions of appropriate units of area unit, including recognition of inverse relation between unit and area magnitudes.	Chooses b and explains that each square centimeter covers less space and so it takes more of them to cover the area of the rectangle.	

Blackline Masters for Rectangles A, B, & C — Area Unit 1



Blackline Masters for Rectangles A, B, & C