Unit 4

Personal Unit Tape Measure

This unit revisits concepts of measurement but in this lesson, the measurement units are not tied to feet, and students are introduced to fractions as partial-units. The goal for students is to create a tape measure composed of "personal" units, and to measure the length of different objects with this personal tape measure. A personal unit is a unit named after each student (e.g., 1 Nina). Personal units are rectangular strips ranging from 1 by 4 inches to 1 x 16 inches. (The long side of the rectangle is what is used to measure length, but its 2-D structure helps students better visualize the results of folds.) The teacher assigns each student one type or personal unit length (perhaps by lottery). Classroom discussion focuses on splitting units, so that lengths that are not multiples of whole numbers can be measured. Fractions, a/b, are quantities representing *a* copies of *b* congruent partitions of the unit. The symbolization a/b corresponds to partitioning the personal unit into b congruent partitions by folding, and then traveling (walking) *a* of these partitions, starting at the zero. Hence, 1/4 unit represents traveling from the origin, 0, to the end of the first of 4 equal partitions of the unit. Similarly, 3/4 unit represents traveling from the origin to the end of the 3rd of 4 congruent partitions, and 5/4 unit represents traveling from the origin to the end of 5 of these congruent partitions, each of which is 1/4 unit long. An alternative interpretation of a/b is *a* iterations of 1/bth partition. Hence, 2/4 is 2 iterations of 1/4th.

Materials

- Adding machine tape
- Felt-tip markers
- Glue sticks
- A giant foot (a strip that is 2 feet-long units)
- Identical units: 5 construction paper strips per student and extras for teacher demonstration

Lengths of strips in inches:

1 x 4
1 x 8
1 x 16

(More proficient students should use the shortest units).

In this lesson. . .

Part One: Introducing the Unit – Each student will make a "personal unit tape measure" with 5 personal units, but this time, the goal is to create parts of units so that the measurement will be more accurate.

Part Two: Partitioning Problems – Students work in pairs to solve two different partitioning problems involving one half and then one quarter.

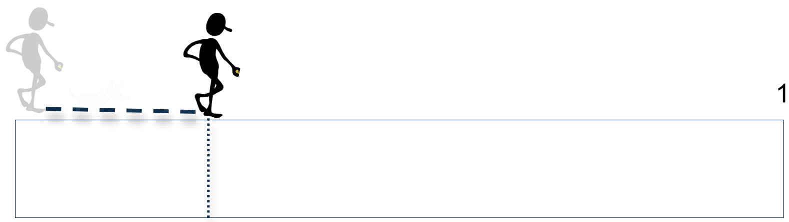
Part Three: Constructing the Tape Measure – Students construct tape measures, label them, and then give them to other students to measure lengths selected by the teacher.

Part Four: What Have We Learned? – An instructional conversation reflects on the problem of how to name partial units.

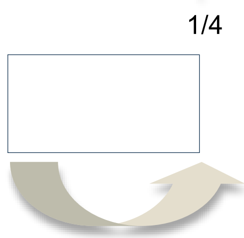
Teacher Note

There are two metaphors used to help students understand fractions. One relies on unit iteration, where, for example, 3/4 means 3 iterations of 1/4 unit. This is called a copy of a partitioned unit, as in 3 copies of 1/4 unit results in 3/4 unit. The second metaphor is distance traveled, where, for example, 3/4 means starting at 0 and moving 3/4 unit. Both of these metaphors, one static and the other dynamic, help students form images of fractions that will later help them locate fractions on the number-line, because the number-line is an idealized ruler.

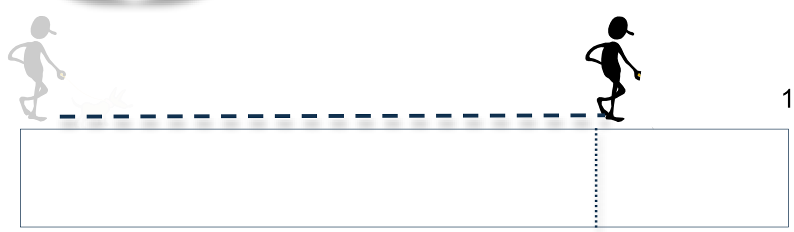
Distance traveled – moving 1/4 unit:



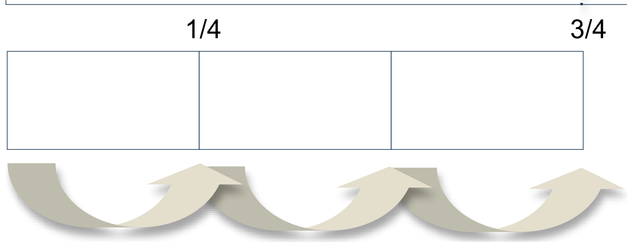
1/4 unit iterated once:



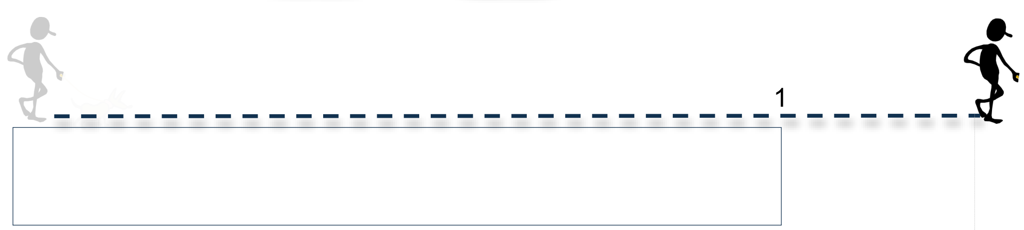
Distance traveled – moving 3/4 unit:



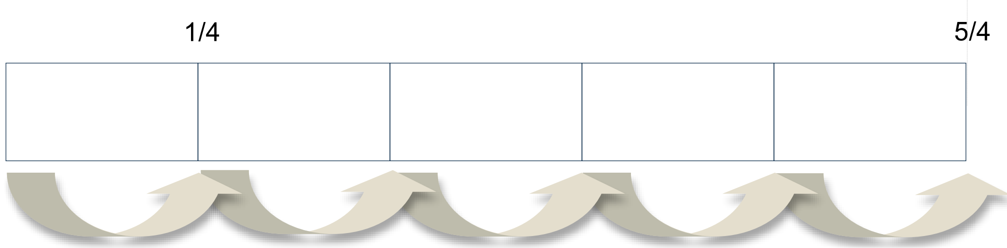
1/4 unit iterated three times:



Distance traveled – moving 5/4 unit:



1/4 unit iterated five times:



Part One: Introducing the Unit

1 – Whole-group discussion about partial units → **2 –** Teacher and student demonstration of partitioning → **3 –** Counter example and relational thinking → **4 –** Writing fractions

Task: Students are introduced to personal units and discuss possible reasons for partitioning units. What does partitioning help accomplish?

1 Teacher introduces "personal units" and explains that each student will get a set of his or her own personal units. Each student will make a "personal unit tape measure" using 5 of his/her units and use it to measure the length of a few different objects. Before constructing the tape measure, the teacher facilitates a discussion about issues students had with their footstrip tapes, focusing on issues with accuracy and problem solving around measurements that were partial unit lengths.

Grounding the Need for Partial Units

The teacher might lead off with: When we used our footstrip rulers to measure, a number of you mentioned that some of the things you were measuring did not come to the end of a unit. (Have several students' footstrip rulers handy for demonstration purposes.)

Ask:

Who can remind us of that conversation?

What problems did you have?

What did you do when the object you were measuring did not end where a unit ended?

What did you have to think about?

Teacher Note

It is often helpful to have a 2-foot strip named after the teacher (the teacher's personal unit). Ask students for their help in measuring an object about 1/2 times as long as the teacher personal unit. The visual perception of the distance makes the need for a part-unit clear.

Part One: Introducing the Unit (continued)

2 Teacher and Student Demonstration of Partitioning

How could I split my unit so that I could measure exactly something that was one-half times as long as this unit? Most often, students will suggest folding it. If they do not, proceed to demonstrate. How should I fold my unit to make 2 parts, each exactly the same length? Tell me what I should do. Demonstrate splitting the unit by folding into 2 congruent pieces.

Ask:

How do I split a unit to make sure the parts are the same (identical, congruent) length?

What do I call each part? [One-half <unit name>, e.g., one-half Nina; **be sure to mention the unit!**]

How do you know it's really a half?

How far have I traveled along my unit?

What do I call the length from here *(start)* to here *(midpoint)*? [Move fingers along the paper strip. Students will likely say "one-half."]

One half of what? [One half of <personal unit name>]

Student Demonstration

OK, now you try it, using your personal unit.

After each student folds the paper in half, ask students to touch one end of the strip and to close their eyes and move their finger until they have traveled 1/2 unit.

Teacher Note

This builds on a resource that most children use—they know about "half of." Emphasize that each partition is congruent (by folding) that there are 2 partitions, and that the measure of the length is now 2 halves (2 split-units, each is one half the distance traveled of the whole unit). 1/2 means 1 copy of the partitioned unit. It could be called anything but we usually call it a half to remind ourselves of its relation to the original unit—1/2 foot or 1/2 <personal unit name>. Sometimes, we choose to lose this relationship. For example, 1/12 of a foot is called an inch. But we want to keep the relation squarely in front of children, so do not rename the partial unit.

Part One: Introducing the Unit (continued)

3 Counter Example

Fold the paper strip into 2 non-congruent partitions in a way that makes their difference in length very clear.

Ask: Can I call from here *(start at 0)* to here *(move finger until you touch the crease-line)* one-half <unit name>? Why or why not?

Give each student another personal unit and ask each to fold one in a way that does not create a half but where there are 2 partitions. Ask students to draw and write in their journals why one of the fold lines creates halves but the other does not.

Teacher Note

Explicit conversation about equi-partitioning is extremely important, so that students understand that not any partition will do. It is important to emphasize too that the distance traveled is identical, so that students do not simply think of equi-partitioning as matching the areas of the strips (because we are considering partitions of lengths). However, the congruence of areas is a good resource to help children see equi-partitioning, so encourage that view as well. Just supplement it with talk about same distance traveled.

Part One: Introducing the Unit (continued)

Building Relational Thinking

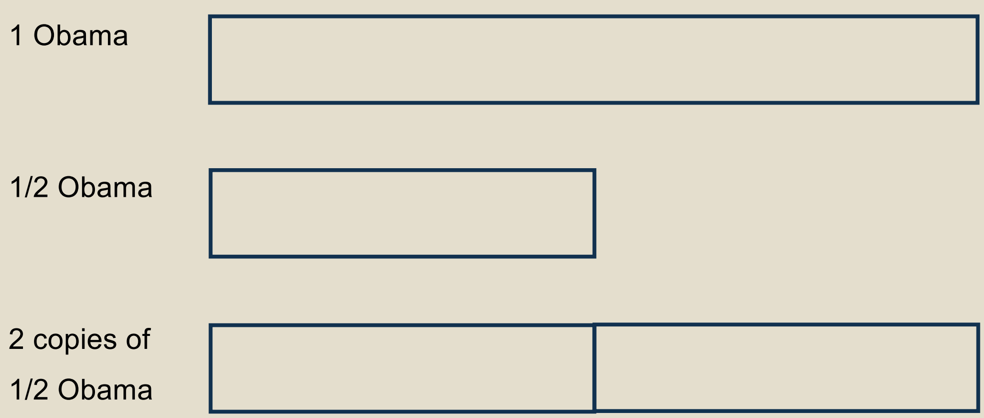
Ask:

1 <unit name> is how many times as long as one-half <unit name>?

How many times would you have to copy 1/2 <unit name> to make 1 <unit name>?

Teacher Note

The construction "times as long" may seem awkward, but it is a gateway to building reasoning about relations. This is often called algebraic reasoning. So, please stick with it. If your class seems to be grasping these ideas quickly, you might ask: How many times would I have to copy 1/2 <unit name> to make 2 <unit name>? Keep the unit and the copy of the unit separated, so that students do not think that we need only make 1 copy of 1/2 unit to make 1.



Write:

1 <unit name> is 2 times as long as one-half <unit name>

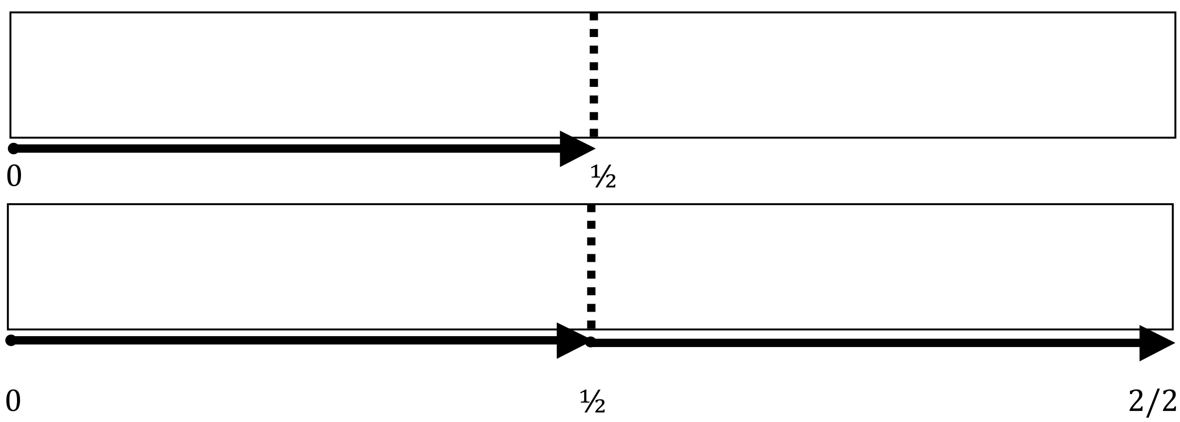
2 <unit name> is 4 times as long as one-half <unit name>

Demonstrate use of iteration to check on the relation (move the half-unit 2 or 4 times). Let students try it out with their personal units.

Part One: Introducing the Unit (continued)

4 Writing Fractions: How might we write one-half with numbers?

Here is how we write so that other people know what we mean: 1/2. The bottom number tells us how we have *split* our unit—how many congruent parts. *Congruence means that the parts match exactly, so that when we put one length on top of another, the distance is exactly the same.* The top number tells us how many copies of these parts that we have. So, if we have 1/2, it means that we have 1 copy of the unit that we have split into two congruent parts, so that it should take 2 of them to make one unit. Review the previous demonstration by iterating 1/2 twice to make 1 unit. Then emphasize travel: So, when we travel 1/2, we start at zero and travel all the way to the end of the first part (*Figure 4.1 below)*. If we keep traveling, and travel all the way to the end of the second part, we write 2/2. This means that we have traveled 2 of the 1/2 units.



Ask: Thinking about this 1/2 unit, what does 2/2 mean? How many? Of what? What about 3/2? Using your personal unit, close your eyes and travel from the beginning to 1/2 unit. Now travel another 1/2 unit. How far have you come? What about another 1/2 unit? How far would that be altogether?

Teacher Note

Establish a relation between *(for example)* a measure of 4 units as 4 iterations of a single unit and 3/2 as 3 iterations or copies of 1/2 unit. 1/2 means 1 copy of 1/2, 2/2 means 2 copies of 1/2, 3/2 means 3 copies of 1/2, 4/2 means 4 copies of 1/2. We want children to get comfortable thinking about measurement as iteration of a unit, but at the same time, we want children to start to think about "times as long" to consider relations between splits of units and the original unit. Language such as 3/2 unit is 3 times as long as 1/2 unit, and 3/2 unit is 1 1/2 times as long as 1 unit helps students think about these relations.

Part Two: Partitioning Problems

1 – Problem 1, partner work and whole group discussion → **2 –** Problem 2, partner work and whole group discussion → **3 –** Building Equivalence

Task: Students solve two problems using their personal units as a way to begin thinking about partitioning.

1 Partner Work and Whole-Group Discussion, Problem 1

Students work in partners to solve problem 1. Teacher facilitates discussion about findings before students move back to partner work to solve problem 2. Teacher should rove the partners to support thinking about strategies for problems 1 & 2. Take notes about student difficulties. Try to get each pair of students to see how they are thinking about fractional parts and labeling those parts, without giving direct instructions.

Problem 1: How could I make 1/4?

Ask: How could we fold so the result is 1/4 of 1 unit?

What would we need to think about if we wanted to find a length that is 1/4 of 1 unit?

How would we know it is 1/4?

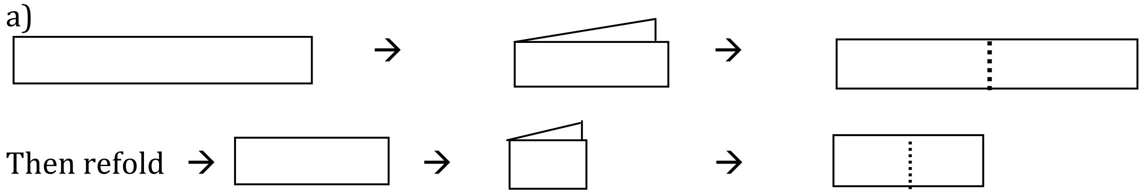
Discuss these questions with your partner and then use paper strips to test your ideas. Make sure you can demonstrate your thinking so you can convince your classmates that you found 1/4 of 1 unit. Record your thinking in your math journal and be ready to explain your thinking.

Teacher Note

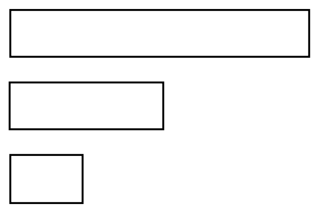
This problem is challenging because it asks students to compose splits of a unit: 1/2 of 1/2 (of 1 unit). Let students work in pairs to answer the first problem, then return to the whole group. Compare solution strategies (See discussion questions below). One common strategy is to fold the paper in half twice. Be sure to allow children to perform this folding. After the first fold, ask children to name the part of the unit. Review the notion that the unit has been split into two congruent pieces. Then, refold the unit, and ask children to fold the 1/2 by 1/2 again. Is it often helpful to tape these successive actions on the unit strip to the board, so that students can see 1 unit (it is always important that students see 1 unit), 1/2 unit, and 1/4 unit. Ask student to unfold their strips and count the number or partitions. Ask how they might say and write the distance traveled between the beginning and end of one of the 4 partitions (1/4 or "one-fourth." Some students may not be challenged by this problem or may finish it quickly. If so, ask them to try to create 1/8 unit.

Part Two: Partitioning Problems (continued)

a)



b)



1 unit

½ of 1 unit or ½ unit

½ of ½ of 1 unit or ¼ unit

Discussion

Ask students to display their results for finding 1/4 of 1 unit. Post the strips that students offer.

Why do you think this shows 1/4 of 1 unit?

How far is it from here (begin at 0) to here (first fold line should be 1/4)?

How do you know for sure?

How could we test it?

How did you create the 1/4 unit?

How many copies of 1/4 unit are needed to make 1 unit?

How can you tell? (Connect iterating 1/4 unit 4 times with 4 copies of 1/4 unit)

Teacher Note

Students may have used two consecutive vertical folds to find 1/4, or they may have made 1 vertical split and 1 horizontal split. If this happens, be prepared to compare and discuss their 2 different methods of folding and the difference in results. Make sure to ask which 1/4 unit would best serve measuring distance traveled. Students may also present a unit that has been split twice, but they may open it up. This will call for a discussion how much of the unit is showing and what we could call it (4/4 or 1 unit split into 4 equal lengths). Then compare it to the unit folded to show 1/4.

How would we write 1/4?

Why would we write 1/4 that way?

What does each part of the symbol mean?

Part Two: Partitioning Problems (continued)

Teacher Note

Be sure the symbolize this as 1/2 of 1/2 of 1 <unit name> (as well as 1/4) to emphasize that the symbolism captures what is essential about the activity—even though the result is a different length for differing personal units, the result is the same: 1 is now measured in 4 of these new sub-units. Emphasize again that the part-of-the-unit traveled is one-fourth of the length of the unit and that the unit is four times as long as the part-unit. If students are finding this well within their grasp, ask them to compare 2/4 unit to 1 unit: 1 unit is 2 times as long as 2/4 unit.

Building Relational Thinking

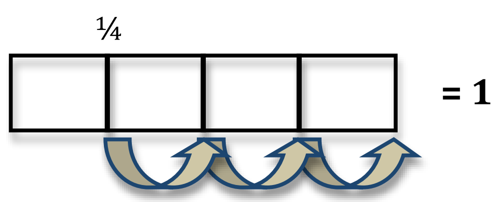
How many copies of 1/4 <unit name> do we need to make 1 <unit name>?

4 of 1/4 <unit name> is 1 <unit name>.

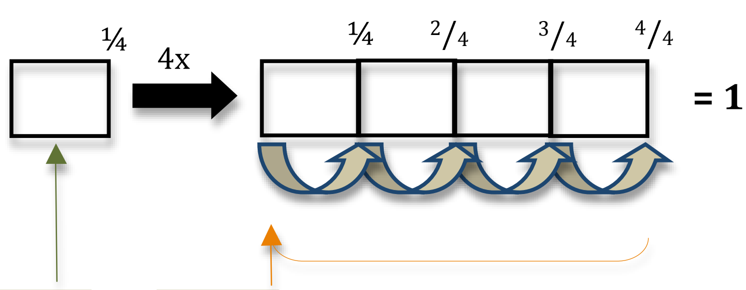
1 <unit name> is ________ times as long as 1/4 <unit name>

Teacher Note

Another way to emphasize the reciprocal relationship between the part-unit and the original unit-length is to ask students to consider how they might recreate the unit-length using the partitioned unit. For example, 1 unit can be recreated by iterating the 1/2 unit twice; 1 unit can also be recreated by iterating the 1/4 unit 4 times. It is important too that children see that there are 4 iterations of 1/4 to recreate 1 unit. Some may say that it only takes 3 iterations, because they think that:



instead of:



To forestall this tendency, keep **the unit** and its copies literally separate.

Part Two: Partitioning Problems (continued)

Building Equivalence

Travel 1/4 unit. Now travel another 1/4 unit. How far have you traveled?
Travel 1/2 unit. How far have you traveled?
How is 2/4 unit the same as 1/2 unit? How is it different?

When we travel the same distance from the beginning (we land at the same place), we call these distances equivalent. So, 1 copy of the unit split into 2 partitions (1/2) means the same thing as 2 copies of the unit split into 4 partitions (2/4).

Teacher Note

For the same unit lengths, equivalence means the same distance traveled, although the split-units might be different. That is, 1/2 <unit name> vs. traveling 2 of 1/4 <unit name>. Clearly, 1/2 and 1/4 are not identical because they represent different partitions of the unit. This lack of identity is confusing to some children, who think that "equal" is a synonym for "exactly the same."

Part Three: Constructing the Tape Measure

1 – Individual work: construct tape measure of personal units → **2 –** Whole-class conversation about comparison of measurements

Task: Students use personal units to construct a personal tape measure that indicates partitions. They use their rulers to measure the length of various objects, such as the circumference of a beach ball and the length of a chalkboard.

1 Individual Work

Have students construct tape measure using 5 of their personal units. They must decide how to partition, by folding, the unit and how to mark each partition.

> **Teacher Note**
>
> For students with higher skills, the partitions can be made more challenging by considering 8th's or 16th's. You can pose this problem by asking students to anticipate what would happen if the 1/4 strip were again split or folded in half again (8th's) or even two more times (16th's). Recall that some students may also be using the shorter personal units, so if they decide to use 16th's, a longer personal unit might be easier to fold into 16th's or 8th's.

Before construction, ask:

What will you need to think about as you construct your measuring tape? Remember, you will use this measuring tape for measurement tasks later.

How will you decide how to partition your personal units?

What will you need to think about when you label (write the different lengths) your measuring tape?

Part Three: Constructing the Tape Measure (continued)

Say:

Remember when you are labeling, the labels show how far you have traveled.

(It could be helpful here to have on hand a copy of those two rulers they used to drag their finger along in the last lesson, just as a reminder – the rulers with labels in the middle vs. at the end.)

Here's my personal unit.

(Have folded into fourths. It is helpful if the unit is much longer than any of the other personal units. It makes clear that not all 1/4 units are the same length. The length of the partition depends upon the length of the unit.)

What do I call this distance, where I haven't traveled at all—where I haven't even started traveling?

Move your fingers along to the 1/4 unit. How far have you traveled?

(Make sure the students refer to the unit name at the end of their answer when referring to teacher's unit.)

Pull your hand along to the 1/2. How far have I traveled now?

(Children may respond with 1/2 or 2/4. They may also say that you've traveled another 1/4. Clarify that you want to know how far you have traveled from the beginning of the unit, then have a brief discussion about the different names for the distance traveled.)

Which number will you use to label that length? Why?

(It is fine to label with both numbers. Ask children why they might write 2/4 and 1/2).

Continue on with this line of questioning for the rest of the partitions of the unit.

Part Three: Constructing the Tape Measure (continued)

Teacher Role

As students construct their tape measures, probe their understanding of how to label the partial units.

What do students call each partition?

We want to ensure that students understand that each partition is called 1/b, where b corresponds to the number of congruent partitions of the unit.

Where do they write the numeric label?

This is often particularly revealing. Students who do not label the unit at its endpoint may not have a firm grasp on the measure as a distance traveled.

Here's my personal unit (have folded into fourths). **What do I call this distance—where I haven't traveled at all, where I haven't even started traveling?**

Pull your hand along to the 1/4. How far have you traveled?

Although this has been discussed in whole group, it is helpful to check on the progress of individuals.

After students have constructed their tape measures, be sure that they try to use them to measure a length that is *less than* the length of the unit, and a length that is *greater than* 5 of the units.

Part Three: Constructing the Tape Measure (continued)

2 **Whole-Class Conversation**

Compare the measure of the length of the same object using different personal units as measures. Why are the measures not the same?

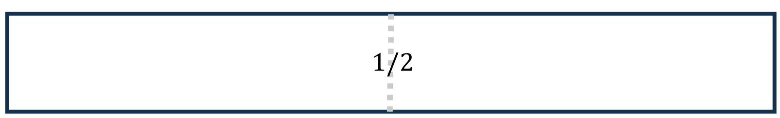
This provides an opportunity to consider the relation between the length of the unit and the measure of the length of the object (i.e., the inverse relation between the length of the unit and the measure of the length, because fewer units are needed to cover the same distance if the length of each unit is longer **when compared to** other, shorter units.

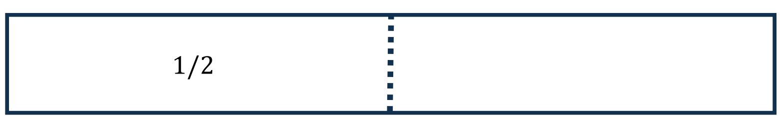
What would make comparison of the measurements easier?

This is the role of a *standard unit*. To make this point clearly, select one of the personal units as a standard. The commonly used standards are no less arbitrary.

Where do we label our units?

Compare labels at the endpoints to labels in the middle of the unit. If this comparison is not available in the work of the students, then create it by drawing two different approaches, as shown in the diagram below:





Part Four: What Have We Learned?

Individual Student Writing (Assessment)

Ask students to spend approximately 15 minutes writing a reflection in their math journals telling how their thinking changed from the foot-strip tape to the personal unit tape. Suggest that students have both rulers on their desk as they write the reflection. Tell students you will choose several journal entries to be read to the whole group during the next measurement lesson. Allow students to write unprompted, as this is a nice opportunity to see the topic on which the students focus.

Instructions

Write complete sentences telling what you know about measurement. After you finish your reflection, go back and read your previous measurement entries, then make any necessary additions to your reflection.

- How is your thinking the same?
- How has your thinking about measurement changed from your previous journal entries?
- What questions do you still have about measurement?
- Include in your reflection: Where do you start measuring? What do you call it?

Suggested writing prompts (if needed):

- What was the same about measuring with the footstrips and measuring with the strips we used today? What was different?
- What have you learned about measurement that you did not kow when we first started measuring?
- If we all used exactly the same personal unit, why might that be helpful?
- What questions do you still have about measuring?

Teacher Reflection and Preparation

Read through the entries before the next class and choose 3 to share. Write a summary of your three focus students:

- How are your students thinking about fractional lengths?
- What surprised you?
- What, if anything, did they find difficult?
- Do students understand the advantages of using a standard unit (e.g., if everyone used the same personal unit, it is easy to compare measurements)?

Appendix

Performance Assessment – Instructions

Performance Assessment – Teacher Notes

Performance Evaluation

This assessment task is designed as a 1:1 performance evaluation. It should take no longer than 2-3 minutes per student (with a minute or 2 between to write observation notes) as it is an assessment and not an instructional opportunity.

Assessment

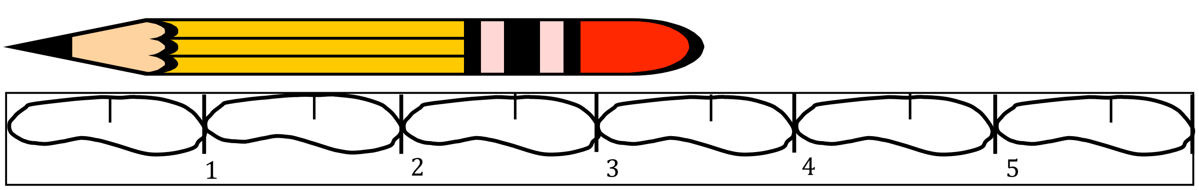
A common challenge is how to label various distances traveled, such as 1 ½ units and 3 ¾ units. Students will be asked to use their personal measuring tapes to measure and write the measurements of 2 objects as you look on. The first object will need to be 1 ½ personal units long and 3 ¾ personal units long.

Preparation

Since student's will be working with three different lengths of units (1 x 4, 1 x 8, and 1 x 16), you will need to either find 3 sets of object that correspond to the 2 lengths on students rulers (1 ½ units and 3 ¾ units), or you will need to build your own measuring tape, leaving all labels off, and have 2 objects that are 1 ½ and 3 ¾ units long when measured with your measuring tape. Measurement and a sample assessment sheet follow:

Personal Tape Measures	Personal Unit Measure	Necessary Object Length
Group A Personal unit = 1" x 4"	1 ½	6"
	3 ¾	15"
Group B Personal unit = 1" x 8"	1 ½	12"
	3 ¾	30"
Group C Personal unit = 1" x 16"	1 ½	24"
	3 ¾	56"

Zero Point: Give students a footstrip ruler with 1/2-unit markings. Have students use their ruler to measure something that ends at one of the 1/2-unit markings. What do they call the measure? Some students who are thinking about units as labels and not lengths may treat 3 ½ as 4 ½ because they think: "1, 2, 3, 4" because it is the fourth count, and 1/2 because it is half of the fourth unit *(see figure below)*.



Measurement Assessment 4.1

Name ____________________________________ **Date** _______________________________

Object 1 __________________________________ **Length** _____________________________

Object 2 __________________________________ **Length** _____________________________

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Teacher Notes: