Pumpkins, Pumpkins! Foundations of Measure

Unit

1

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Mathematical Concepts

- Objects are described by their attributes, such as length and weight.
- Attributes must be defined if they are to be used for comparisons.
- Attributes of objects, such as height, can be compared directly.
- Attributes of objects can be compared via representations of attributes.
- Objects can be ordered by the magnitude of one or more attributes.
- Agreements about methods of comparison facilitate object comparisons.

Unit Overview

This unit introduces students to foundations of measurement. Students look at a few pumpkins and tell what they notice about them. Students typically notice attributes such as texture (smooth, bumpy), size (big, little), girth (fat-around), color, and height. What children notice establishes that an object like a pumpkin can be described by multiple attributes. The teacher asks which of these attributes or qualities will help determine the "biggest" pumpkin. Different senses of "bigness" are elicited from children, and the teacher guides consideration of two attributes: height and girth (circumference). What is meant by height? By "fatness?" Because girths are difficult to compare directly, representations are introduced to facilitate indirect comparisons. Using the representations, children order the pumpkins, first by girth (one order) and then by height (a different order). They learn that agreeing on common methods of measure is necessary to reduce variability from student to student in the representations of girth or of height of the same pumpkin, and that agreed-upon means and methods of representation facilitate reliable orderings of pumpkins by girth or height. The unit concludes with a formative assessment involving comparison of a pumpkin that students can see to another hidden behind a screen.

Which is Biggest? (First Attempt)

The teacher asks students to determine which pumpkin is biggest by comparing girth or height. Although students are permitted to make comparisons directly (by moving pumpkins), the teacher introduces the representation of streamers to support indirect comparisons, especially of

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girth. Using streamers, small groups of students develop methods to order the pumpkins either by girth or by height, but it is likely that their methods will vary from group to group. Students post their streamers on chart paper to permit public inspection of the results.

Looking at the posters, students propose reasons for the variability in their representations of the length or girth of the same pumpkin. They refine their methods to produce greater agreement.

Which is Biggest? (Second Attempt)

Students work in small groups with their refined methods and again post their representations, along with their orderings of the pumpkins' girth or height.

Formative Assessment

The lesson concludes with a formative assessment that requires comparing a visible pumpkin to a mystery pumpkin that is screened from view.

Extension Activity

Using a pan balance, children order the pumpkins from lightest to heaviest.

Materials & Preparation

Read

- ☐ **Theory of Length Measure (ToM) Construct Map**
- ☐ **Unit 1**

Start by reading the unit to learn the content and become familiar with the activities.

Academic Vocabulary

Height
Circumference
Weight

Gather

- ☐ 4 pumpkins (different sizes and shapes, so that ordering by girth is not the same as ordering by height)
- ☐ Chart paper
- ☐ Streamers (4 colors, one for each pumpkin)
- ☐ Tape
- ☐ Student journals
- ☐ Teacher math journal and/or flip camera to document the lesson

Prepare

The pumpkin measurement requires a large open space. Identify a space (playground, field, gym, a part of the classroom) large enough so that students can measure the pumpkins. Groups should be spread apart to avoid sharing information; each group explores in its own way. This is likely to produce variability in the streamer lengths for the same pumpkin. Accounting for and proposing ways to reduce this variability are important, so it is important to preserve the independence of the groups.

Mathematical Background

The big ideas of this unit are (a) defining attributes of an object, (b) using representations—streamers that stand in for height and girth of a pumpkin—to allow direct comparison of pumpkins via their representations, and (c) employing common methods of representation to facilitate comparison. The mathematical foundations are described in the Theory of Measure construct map.

Mathematical Literacy

Children develop academic vocabulary: height, circumference, and weight. They write about their findings in their journals. They engage in conversation about methods for comparing pumpkin girth and height.

Which is Biggest? (First Attempt)

Students notice attributes of four different pumpkins labeled A, B, C, and D (the "pumpkin patch"), and establish a way to use a representation to stand in for girth and height.

Which is Biggest? 1
Which is Biggest? 2
Formative Assessment
Extension Activity

Whole Group

1. **With a KWL chart, the streamers, and four pumpkins, ask students: "What do you notice about the pumpkins?"**

    Students will typically notice a variety of qualities of the pumpkins, such as color, size (e.g., big, small, skinny), shape (e.g., round, oval), and texture (e.g., bumpiness).

Construct: ToML 1A, 1B and ToML 2A, 2B, 2C
This task engages students in thinking at the early levels of the Theory of Measure construct.

2. **Which of these (what students noticed) can help us find out which pumpkin is biggest? What could we mean by biggest?**

    Students typically suggest that the biggest pumpkin is the tallest (height) or the fattest (circumference). When students suggest ideas like these, teachers suggest academic language: height and circumference.

3. **How can we find out which one is the tallest?**

    Students often want to directly compare the pumpkins. Allow them to do so. Then introduce tape or streamer and ask if the streamer could be used to find the tallest. What would we have to do?

4. **How can we find out which one is the fattest? Could we use the streamers here, too? How?**

    Students will typically use a streamer to wrap around the pumpkin but they may choose different senses of "fat." Some may look for the longest circumference, but others may be inconsistent in where they wrap each pumpkin.

Which is Biggest? 1
Which is Biggest? 2
Formative Assessment
Extension Activity

Small Groups

5. **Students are divided into four groups. Each group measures the height of circumference of four different pumpkins to order them from smallest to biggest.**

   a. Review KWL chart. Ask: How can we use the streamer to measure the pumpkins? Students will typically respond "wrap around" (circumference) or "hold it up" (height). Teacher records answers on the "W" of the KWL chart.

   b. Explain job titles:

      - pumpkin holder (holds pumpkin steady while streamer holder measures)
      - streamer holder (measures pumpkin)
      - streamer cutter (cuts streamer after the pumpkin has been measured)
      - streamer taper (tapes the streamer onto the data chart)
      - measurement drawer (draws a line on the data chart to show which way the pumpkin was measured—i.e., a horizontal line for circumference and a vertical line for height)

   c. Explain that you expect productive group work and what you think that will look and sound like. Each group will rotate to each pumpkin and perform their job duties.

   d. Each group will use the streamers to propose a pumpkin order from smallest to largest. If students ask whether big means height or circumference, tell them it is up to them. Each group records its findings by pasting the streamers on a chart labeled A, B, C, or D, so that someone else will know which pumpkin was biggest, which was smallest, and which were in-between.

Whole Group

6. **Display the data posters, one from each group.**

Construct: ToML 2C, 2D

Choose a poster that provides opportunities to consider ideas such as the need to compare all the pumpkins on a common attribute (i.e., height or circumference) or the need to compare congruency of the streamers from a common starting point. Some posters may mix attributes or fail to align the streamers for purposes of comparison.

7. **The teacher asks students to tell about the poster, and using response from students:**

    a. Streamer placement: How does this affect our findings? (If the streamers don't all start at the same place on the poster, how does that affect our comparisons?

    b. If students did not compare all of the pumpkins on the same attribute (mixed height and circumference): How does this affect our findings?

8. **For each pumpkin (A-D), transfer all the groups' streamers that represent height onto another data poster, using a common starting point. Do the same thing for all the streamers representing circumference.**

    a. For each attribute, ask:
    What do you notice? *Students should notice differences in lengths.*
    What could we do differently next time so our streamers for height will be more alike?
    What could we do differently next time so our streamers for circumference will be more alike?

    b. Teacher records response from students onto the "L" of the KWL chart.

Which is Biggest? (Second Attempt)

Which is Biggest? 1
Which is Biggest? 2
Formative Assessment
Extension Activity

Students re-represent the pumpkins, labeled A, B, C, and D, and order them from smallest to biggest.

Whole Group

Construct: ToML 2C, 2D

1. **Review the KWL chart, job titles, and expectations for productive group work. Each group will measure each pumpkin and record the findings.**

2. **Display posters with streamers representing height or circumference.**

   Do all the groups agree about the ordering of the pumpkins by height? By circumference?

3. **For each pumpkin, transfer all the streamers representing height onto another data poster, using a common starting point. Do the same thing for the streamers representing circumference.**

   Q: Do the lengths of our streamers for **height** tend to agree more this time? Why or why not? Were the streamers for some pumpkins more the same than for others?
   Q: Did the lengths of the streamers for **circumference** tend to agree more this time? Why or why not? Were the streamers for some pumpkins more the same than for others?

4. **Record additional findings on the "L" of the KWL chart and review KWL chart.**

Which is Biggest? 1
Which is Biggest? 2
Formative Assessment
Extension Activity

Formative Assessment

Students compare one of the patch pumpkins to a hidden pumpkin.

Whole Group

1. Display the streamers representing the height and circumference of the patch pumpkins.

a. Remind students which streamers represent height and which represent circumference.
b. Say: We have a hidden pumpkin. Your job is to find out if the hidden pumpkin is bigger than this one (show class one of the patch pumpkins). You can take whatever tools you like with you to help you make that decision (provide ruler, streamers, etc.).

Individual

2. One at a time, students will go to the hidden pumpkin.

a. Students will choose a measurement tool and measure the pumpkin.
b. Students will tell the teacher which pumpkin is bigger and say how they know.
c. Teacher will record the answers on the Formative Assessment record provided (teacher looks for use of tools, whether or not child compares the class pumpkin and the hidden pumpkin by aligning the streamers, use of attribute words of height or circumference).
d. Students write in their journals about what they learned about comparing pumpkins.

Which is Biggest? 1
Which is Biggest? 2
Formative Assessment
Extension Activity

Extension Activity

Students order pumpkins by weight with a pan balance.

Whole Group

1. **Introduce students to the pan balance.**

   Place an object in one pan. Ask students to predict what will happen if a second object is placed in the other pan and why they think so.

2. **After establishing that the heavier object goes down, ask students how they would record that one object is heavier than another, and how they could write that.**

3. **Ask students to predict the order of the pumpkins in the pumpkin patch, from lightest to heaviest.**

4. **Use the pan balance to establish the ordering of pumpkins by weight. Compare this ordering to those of height and girth (circumference).**

   Ask students: If we know that A is heavier than B, and B is heavier than C, do we need to compare A and C with the pan balance, or can we already tell which is heavier? This transitive inference may be challenging to students, but it is worth trying to get students to think about it. If they say that the pan balance must be used, then go ahead and use it.

Formative Assessment

Student ____ Date ______________

Indicate the levels of mastery demonstrated by circling those for which there is clear evidence:

Level	Description	Notes
ToM2A	Identifies attribute for comparing pumpkins. *Circle those that apply:* Height | Circumference | Other ______________	
ToM2C	Child compares class and hidden pumpkins streamers by aligning the ends of the streamers to test congruency and drawing appropriate conclusion.	
ToM2D	Child's method using streamer is that developed in class, so the definition of the attribute is the same for both class and hidden pumpkins.	
ToM1B	Uses streamers but does not align them to ensure adequate comparison.	
NL	Does not use streamers. Makes claim by perception.	

Academic Language: Indicate academic words the student is familiar with by recording them here.

From Here to There (Grade 1-2)

Unit 2

Mathematical Concepts

- Length measure blends two ways of thinking. The first relies on the principles of a ruler. Measure results from repeated iteration of a unit length. The second relies on motion. The measure of a length represents a distance traveled.
- Measure of a length is obtained by iterating a unit and accumulating the number of iterations of that unit without gaps or overlaps.
- The length of a unit and the resulting measure are inversely related.
- The whole distance must be measured.
- There is a need for a standard unit.

Unit Overview

This unit is designed to give students a common measurement experience, finding the distance between two landmarks, in order to support further discussion about the principles of measurement. The landmarks also highlight length as distance traveled. The landmarks are set far enough apart so that choice of a measuring unit has visible consequences. Now that we have an initial assessment of students' prior knowledge about measurement, our goal is to give them some experience measuring that we can use to facilitate their development of big ideas, or principles of measurement, such as the nature of a unit or the need to iterate a unit. This activity is another opportunity to see how students invent measurement procedures, think about unit, and how they talk about these inventions as they discuss things they considered when they measured. We can use what we have learned from the previous assessment to guide our observation of students' activity and to inform instruction.

The results of the measurements give students the opportunity to think further about the relation between the unit measure (the accumulation of units) and the relative length of the unit (longer feet result in smaller quantities). This continues to create the need for standardization—agreeing about sharable units and methods of measure.

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Materials & Preparation

Read

- ☐ **Unit 2**
  Start by reading the unit to learn the content and become familiar with the activities.
- ☐ **Mathematical Background**
  Reread the mathematical background carefully to help you think about the important mathematical characteristics within the unit.
- ☐ **Sample Student Thinking**
  Reread the Student Thinking boxes to anticipate the kinds of ideas and discussions you will likely see during instruction.
- ☐ **Measurement Construct Map**
  Read the construct map and look at the multimedia map to help you recognize the mathematical elements in student thinking, and to order these elements in terms of their level of sophistication.

Gather

- ☐ 2 flags or cones (or find a location with consistent distances for the students to measure)
- ☐ Teacher math journal (for note taking)
- ☐ Comparing Measurement Methods (page 10)
- ☐ Student math journals
- ☐ Chart paper and markers

Prepare

The first part of this lesson requires a large open space. Identify a large enough space (playground, field, gym, hallway) in which students can work to measure the distance between cones, flags or landmarks.

It helps students to think about a purpose for measuring and the importance of getting a specific, repeated measure. It is a critical step to help them make meaning of the need for rules of measure in order to get a consistent measurement otherwise, any measure will do. (i.e. We need to replace a piece of carpet or a pane of glass in our window, or make a sandbox the specified length.) Think about a context related to your school that students could connect to.

Mathematical Background

The big ideas of measurement emphasized in this unit are related to the nature of a unit.

Attribute-unit relation
An object can have multiple attributes, and some units may be more suitable than others for measuring different attributes or even different magnitudes of the same attribute (e.g., a longer distance might be measured more readily with a longer unit, or an area is measured with a 2D unit).

Iteration
A unit is repeatedly applied to obtain a measure. For example, a measure of length is obtained by moving (translating) a unit-distance a finite number of times. The measure is the number of iterations of the unit (e.g., a 9-inch length is measured by translating an inch from the beginning to the end of the length 9 times).

Identity
The units must be identical in order for the iteration to yield a single measure.

Tiling
The units are translated “end-to-end” with no gaps. Units tile the line, plane, volume, etc.

Inverse Relation between Unit Length and Measured Length
For the same length, measuring with shorter units of measure results in a greater numeric measure when compared to the measurement obtained with longer units of measure. For example, a two foot length is measured as 2 feet or as 24 inches, because a foot is twelve times as long as an inch.

Straight
When is a line straight? Often, when we measure a distance between two points, we imagine a line. A traditional way of thinking about a straight line is as the shortest distance between two points. But a way that is more consistent with bodily experience is to consider as straight a line formed by moving without any change in direction. For instance, walking at a constant heading towing a piece of chalk ideally creates a straight line.

Measuring a Distance
Things We Think About

Measuring the Distance Between Two Landmarks

Students use their bodies (such as footsteps) or other tools that they invent (e.g., some think to use notebooks, others pencils) to measure the distance between two landmarks.

Partners

1. **Take students to a large open space such as a playground or a field.**

   a. Set up two flags (or cones) at opposite ends of the space, making sure they are set far enough apart to make measurement challenging, so that different choices of units have consequences that are visible. (This distance can be any location as long as students have a set distance to measure. It is a good idea to have the students spread out so that you get a variety of strategies.)

   b. Ask students to work with their partner to find out how far it is from one flag to the next (or from one landmark to another). Mention they can try out anything they like or use any tool they like, except for rulers. Some students will walk heel-to-toe, others will stride, hop, etc. Some may use non-standard tools, such as pencils, clipboards, and lanyards. It is important to not allow students to use rulers or other standardized measuring tools.

   c. Take notes about what your students do as they measure in your math journal. Look for examples of students applying or not applying the big ideas of linear measurement from page 3.

Individual

2. **Ask each student to record his or her measurement in his or her math journa**l.

   a. Ask students to record in their math journals how they measured the distance. Ask them to include a drawing.

      Note. Some students may draw a picture of themselves enacting a measure, others may choose to be more schematic and just show the unit used.

Teacher Reflection and Planning

3. **Read student summaries and your observation notes about students' measurement methods.** (Select three methods to use in the classroom discussion and record on Comparing Measurement Methods worksheet.)

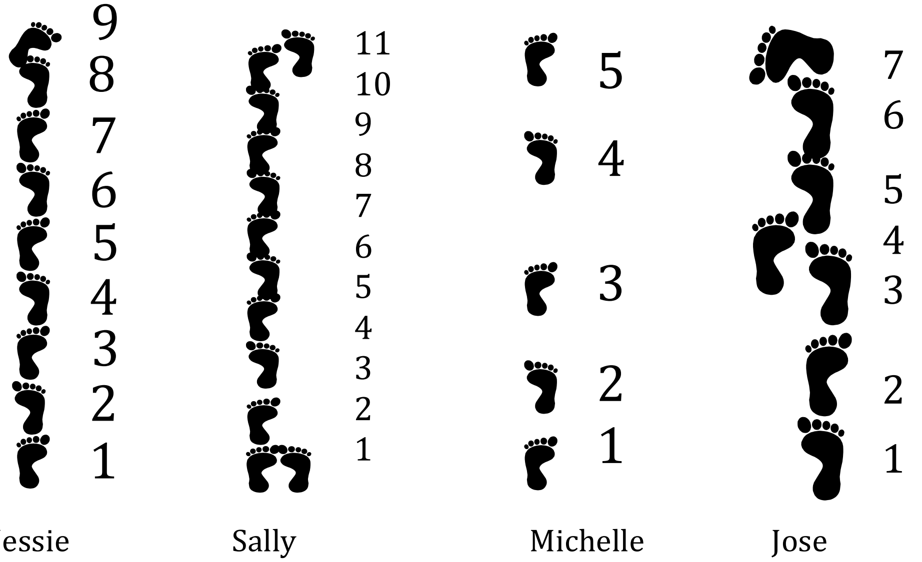
    Note: Think about the students' actions and comments during the activity to help you plan a conversation. (see the Comparing Measurement Methods WS). Questions to guide your planning could include:

    What big idea about measurement do I want to start with (suitable units, identity, iteration, tiling, inverse relation between unit length and measured length, the meaning of straight)?

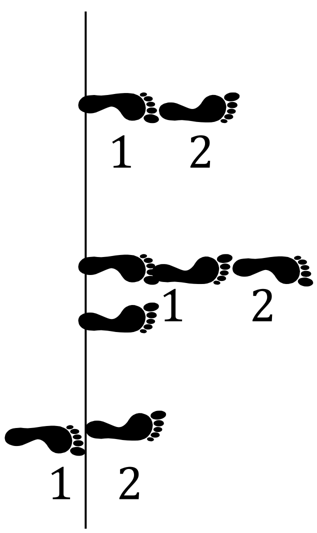
    What did students do that tells you about how they were thinking about identical units and or tiling? (Did they leave gaps? Did their strategies indicate that they value identical sized units? How did they count the last unit if it was a partial unit?

Measuring a Distance
Things We Think About

Student Thinking Some students will think of using their feet. They often do so in ways that reveal their thinking.



Measuring the full length or distance*



Are they counting all the space? Where do they begin their count? Before the measure? Skip the first space? Or do they start where the first footstep begins?

How are they describing the end point? Are they rounding up or partitioning the last unit? (There will be lots of different responses. Keep track of what they are but more emphasis will be placed on this big idea in unit 3.)

- *"Using Children's Understanding of Linear Measure to Inform Instruction" Teachers engaged in Research: Inquiry into Mathematics Practice Grades K-2* (D. S. Mewborn, series ed.). Reston, VA: National Council of Teachers of Mathematics.

Things We Think About When We Measure

Students reason about measurement principles based on their experience with measuring the playground.

Whole Group

1. Elicit Student Ideas about Measuring the Distance.

a. To open up the discussion, ask the class: "What did we think about when we measured?" First, listen to the directions in which students take the conversation and then consider questions you prepared to support further discussion.

b. Conversation starters include:

Q: What did we do?
Q: What were your measures? (Record on board or chart paper.)
Q: Why did we all get different measures?
Q: What were some important things you did when you measured?
Q: Would someone else get the same measurement if they did what you did?
Q: Someone said that they measured the <u>straight</u> distance. What did they mean by straight?

c. Using the planning sheet, ask three pairs to present their methods. As each pair presents their method, ask other students to point out one positive feature of the method and one thing that might be improved so that someone else could use the method. For example, when using their bodies, it might be important for students to tell other students to walk heel-to-toe and to keep moving without changing direction—this will keep the path straight. (It is sometimes helpful for the teacher to enact a method that changes direction, so that this tacit assumption of "straight" can be made explicit.)

d. Following presentation, compare and contrast methods and try to highlight important ideas, such as iteration, tiling, accumulating units by counting the number of iterations, identical units, and the suitability of the unit for the task. Ask:

Q: Could other people do it easily?
Q: What if we used a pencil to measure—would it work? Would you want to be the one using it?
Q: What happens when we measure the distance with a small unit, like a paper clip?

Measuring the Distance
Things We Think About

2. Summary

Use chart paper to summarize what students have learned about what they need to keep in mind when they measure. Be sure to label big ideas, such as "no gaps" and "keep using the same unit"

3. Formative Assessment

The teacher demonstrates using pencils to measure the distance between 2 points that are visible to everyone in the class. She asks children to tell her whenever they disagree with how she is measuring or to write about each mistake that she makes in their journals. Mistakes that a teacher could enact (depending on how the previous conversation went) include leaving gaps between the pencils, using pencils of different lengths, and stopping the measurement when she runs out of pencils.

Comparing Measurement Methods

Summary of what students did:

Learning Goals:

Method/Students:	**Method/Students:**	**Method/Students:**
____________	____________	____________
Ideas to Focus on	Ideas to Focus on	Ideas to Focus on
____________	____________	____________
____________	____________	____________
____________	____________	____________

Connections 1/2:

____
____
____

Connections 2/3:

____
____
____

Connections 1/3:

____
____
____

Using Units to Measure Distances (Grades 1-2)

Unit

3

Mathematical Concepts

- A unit of length is itself a length.
- A length measure results from accumulating iterations of a unit length.
- The length of a unit and the resulting measure are inversely related.
- Partial units are obtained by equi-partitioning of a unit length.
- A measure is the distance traveled from zero. (One unit is only one unit after you have traveled from the beginning of the unit to the end of the unit.)

Unit Overview

Students build a connection between using a body part (the foot) as a unit of measure and a paper strip unit to measure a length. After discussion about which aspect of the foot the paper strip represents (its length from heel-to-toe), students measure pre-selected lengths with the paper strip units. The pre-selected lengths should be:

- Measurable with available paper-strip units with a whole-number result
- Measurable with a whole-number result but without enough paper strips to tile the length (to think about iterating the unit)
- Measurable by splitting one paper strip into 2 equal partitions (a length measured as ½ unit)

As a formative assessment, students predict the measure of the same lengths with a unit paper strip that is 1/4 times as long as the teacher's foot (this relationship is not revealed). Students estimate and then check their estimates. Class discussion focuses on the inverse relation between the measure of length and the length of the unit: Smaller unit lengths result in greater measures. Class discussion should also include a focus on the properties of units, and the partial unit of ½.

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Read

- ☐ **Unit 3**
  Start by reading the unit to learn the content and become familiar with the activities.
- ☐ **Sample Student Thinking**
  Reread the Student Thinking boxes to anticipate the kinds of ideas and discussions you will likely see during instruction.
- ☐ **Measurement Construct Map**
  Read the construct map and look at the multimedia map to help you recognize the mathematical elements in student thinking, and to order these elements in terms of their level of sophistication.

Gather

- ☐ Student math journals
- ☐ Teacher journal for note-taking
- ☐ Post-it notes for marking units
- ☐ Markers
- ☐ Rubber band (one for demonstration, *see page 5*)
- ☐ String (for demonstration, *see page 5*)
- ☐ Construction paper to copy the teacher's footstep cutouts (about 10 sheets), or a few sheets if using paper strips to represent the foot (*see page 7*)

Prepare

- ☐ Space for students to measure a distance heel-to-toe (*see page 3)*
- ☐ Trace and cut out copy of teacher's foot.
- ☐ Construction paper strips longer than the teacher's foot (*see page 5*)
- ☐ If using paper strips, 10 strips (*see page 5*)
- ☐ At least 3 objects for students to measure (*see page 7, paragraph c)*

Using Units to Measure Distances (Grades 1-2)

Mathematical Background

Partial Units
Units can be partitioned to result in partial units (e.g., ½ inch). For example, 1/n unit is obtained by partitioning a unit length, u, into n congruent parts and representing one of these partitions as 1/n. Iterating the 1/n unit n times restores the original unit. There is a significant difference in student reasoning between a student can anticipate this relation and one who must resort to literal enactment to establish it.

Iteration
A unit of length is itself a length. Units can be iterated and the space left by iterating a unit has already been counted and represents the unit of length. A length measure results from accumulating iterations of a unit length.

Inverse Relation between Unit Length and Measured Length
For the same length, measuring with shorter units of measure results in a greater numeric measure when compared to the measurement obtained with longer units of measure. For example, a two foot length is measured as 2 feet or as 24 inches, because a foot is twelve times as long as an inch.

Walking Feet as a Means for Measurement

Students use movement to measure the length of the classroom with their feet and consider the relationship between their actions and resulting measurements. This builds from the work and the conversations the students had in Unit 2.

Walking Feet and Measurement
Units Represent Feet
Using Units to Measure Distances
Formative Assessment

Individual/Partners

1. **Introduce the task by asking:**

   Q: If we all walk heel-to-toe like this (demonstrate) from here to there, will we all get the same measurement? Students will likely tell you no because everyone does not have the same sized feet. If they do not, you can get two students (with different sized feet and ask if they will both get the same measurement).
   Q: Why do you think so?

2. **What should we keep in mind when we walk?**

   a. Record student recommendations on chart paper.
   b. Have student pairs walk the length of the classroom.

   Note. The teacher should observe while students are walking. Ask students what they are keeping in mind about measurement as they are walking. Is their path straight? How can they tell? What about the foot is helpful for measuring length?

Whole Group

(This will likely be a review to bring to the front the importance of using some type of standard.)

3. **Compare measurements and discuss possible reasons for the different measures, looking for relationships between the method, the unit, and the resulting measure. Ask:**

   Q: What did you get as measures?
   Makes differences among measures visible

Q: We all walked end-to-end and we still don't have the same measure. Why would that be?
Entrée for conversation about the nature of the unit: why do smaller feet result in larger measures?

Q: What did we use to measure?
Invites conversation about the nature of the unit

Q: How did we use our feet to measure? What is important about how we used our feet?
Invites conversation about how students are thinking about iterating a unit

Q: If we wanted to all get the same measurement, what should we do?
This is an opportunity to talk about the value of everyone using the same length as a unit.

Walking Feet and Measurement
Units Represent Feet
Using Units to Measure Distances
Formative Assessment

Units Represent Feet

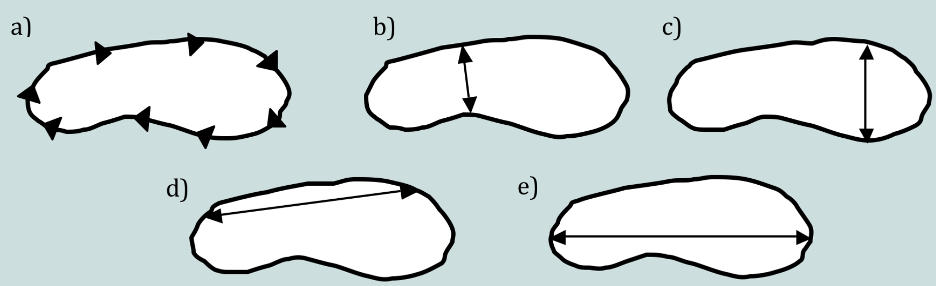
1. **Introduce the teacher's foot unit.**

   Explain that you made a copy of your foot by tracing it, then cutting it out. Ask:

   Q: What about my foot is important for measuring?

Student Thinking

It is important to prompt consideration of which attributes of a foot are being represented by the paper strip or string. For example, feet have curvature along the outline and arches. Does the paper strip represent the length from heel to big toe, from heel to little toe? Width? We want to be specific about what the representation (the cut-out of the foot or the paper strip) is representing and why the representation is useful for measurement of a length.



Above: Possible choices that students might make about what to pay attention to on the foot: a) perimeter, b) & c) width, d) & e) length.

Although some students may not choose (e), don't dismiss the other choices. Instead, try to enact their consequences for the measure of the distance from one end of the room to the other—they each could be used but would require a lot of work to use in that way. For example, you might try enacting choice (c) by sidestepping from one end of the room to the other. Another possibility is to cut out strips of paper and string to correspond to choices (a) – (e) and let children experiment by placing them on the footprint. Which do the best job of representing the distance between heel and toe?

Walking Feet and Measurement
Units Represent Feet
Using Units to Measure Distances
Formative Assessment

2. **Prompt a conversation about the important attributes of the foot cutout. Ask:**
   - Q: What could we substitute for my foot? What would do the same job?
     [Show a construction paper strip that is longer than the teacher's foot cutouts.]
   - Q: Could we use a (rectangular) strip of paper instead?
   - Q: What would we have to do to it so it stands in for my foot? Why?
   - Q: Does it matter how fat or skinny the strip is? Why?

   Note. **Establishing equivalence between a paper strip representation of the foot and the foot is critical.** Units of length measure function like line segments, so it is important that students understand the paper strip stands in for this imagined relation between feet and strips. The aim is to foster representational competence: students need to develop awareness of the purposes and eventually, limitations, of different systems of representation. The strips can represent area as well as length. Hence, it is important to emphasize that length measure involves travel along the edge of the strip. Enacting this aspect of the representation is very helpful. Have students close their eyes and "travel" along the edge of the strip.

   - Q: If we start at one end and travel to the end of the strip, how far have we traveled? (Have students close their eyes and pretend that they are traveling from the start of the unit to the end of the unit.)

   - Q: Why wouldn't this piece of string work as well? (Be sure to kink the string a few times.)

   Note. Make a decision about substituting a paper strip for the foot as a unit of measure. If the class seems to understand how a strip can stand in for a foot, then give each person 5 strips, each of which is as long as the teacher's foot.

Using Units to Measure Distances

Students measure different distances with paper-strip units.

Pairs

Be sure to have several different distances to measure. To reveal student thinking about iteration, tiling and partial units, have students measure distances that are:

- 4 units long (with enough units to measure to reveal ideas about tiling unit lengths)
- 8 units long (without enough units to measure, re-using units for iteration)
- ½ unit long (partial unit distance)
- 2 ½ units long (this is a more difficult problem in that it requires reconciling the number of units and the distance traveled)

Whole Group

4-unit length: Based on your observations, have one or more pairs demonstrate how they measured the 4-unit length. Ask students to consult the earlier chart about important ideas to keep in mind when measuring and to talk about which of these ideas they are seeing at work. You could also demonstrate a measurement of the 4-unit lengths using 5 overlapping units and ask which important idea you did not keep in mind. You could demonstrate the measurement by using 3 units with gaps, again asking children what important ideas about measurement that you had forgotten. Continue the conversation by using 6 units in a saw-shape, asking if the units are being used to measure a straight distance.

Enact traveling: Have students watch as your fingers travel along the paper strips from the starting point to the ending point of the distance. Then re-enact and ask them to use a finger to show 1 when your fingers have traveled ONE teacher-foot. How about TWO teacher-foot? THREE? FOUR?

Set up a contradiction by saying that someone else said that your fingers traveled one about here (gesture to ½ way along the first unit). Ask students to say what the person might have been thinking (it is the first unit) and why the distance traveled is not yet one.

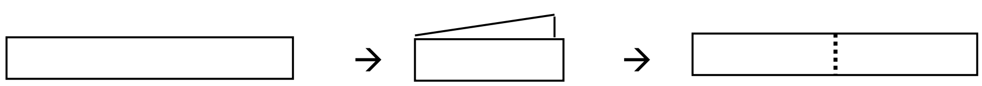
8-unit length: Enact running out of units when measuring the 8-unit length. Based on your observations, ask a pair of students to complete the measurement by re-using the units. Possible questions include:

- Q: Can we reuse a unit?
- Q: Why did you put your finger there (at the 5 unit mark)? What are you trying to keep track of?
- Q: How much of the distance has already been measured?
- Q: How many foot units is this item? How can you know for sure?

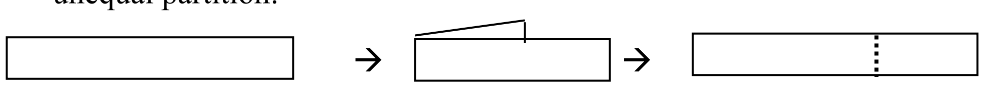
Enact traveling, as before. Be sure to engage students in the measure as a distance traveled, again using fingers to indicate ONE, TWO, etc.

½-unit length: Ask students if the distance is 1 foot, less than 1 foot or more than 1 foot. Align the unit with the distance to provide students with a visual referent.

a. Give each student a foot strip and ask them to fold it so that it is ½ times as long as the foot.



b. Most students will say that there are 2 pieces. Present them with an unequal partition:



c. Ask students to think about what is the same and what is different about the 2 splits. Tell them that one-half foot means that the foot is split into 2 parts that are exactly the same length. We can tell they are exactly the same length because we can put one on top of the other without any leftovers. They both cover one another exactly (they are congruent pieces).

d. Enact traveling. Ask students to unfold the ½ unit strip and to touch one end of the strip. Then ask them to close their eyes and move their finger until they have traveled ½ unit.

2 ½ unit length: If students appear to be comprehending the preceding measures, then ask them to demonstrate the difference between a measure of 3 foot and a measure of 2 ½ foot. Emphasize again the travel metaphor. Have students follow the teacher's fingers as she walks 1 unit, 2 unit, and then ½ more unit for a total of 2 ½ units.

Walking Feet and Measurement
Units Represent Feet
Using Units to Measure Distances
Formative Assessment

Formative Assessment

The formative assessment, located on the next page, indicates student progress in:

a. Representation: What aspect of a foot is represented by a unit of length measure?

b. Partial unit: How can a partial unit, ½, be established by folding (partitioning) a unit length?

c. Conceptions of Units: How can a new unit of length measure be used to re-measure the same distance? (This affords an opportunity to view how children think about iteration and explain why re-measuring with a shorter length unit results in an increase in the measurement—the inverse relation between unit size and measure.

d. Coordinate partial and whole units to measure a length.

After children have completed the assessment, conduct a conversation about each question. The intention of the conversation should be to clarify (a) – (d) above. It is especially important that children understand how units are accumulated to result in a measure, with attention to what happens when we run out of units, the importance of no gaps between units, and the inverse relation between the length of the unit and the measure of the resulting distance. For partial units, it is critical that children understand that ½ refers to an equal partition of the unit length. This can be established by superimposing one part on the other—it fits perfectly. The other way to check is that if a copy of the unit is set aside, it takes 2 ½ units to travel the same distance.

Assessment | Using Distances to Measure Distances **Unit 3**

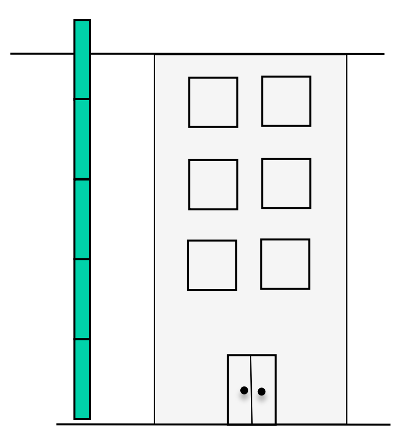
NAME_______________________________________

1. Here is a footprint. If we want to use it to measure how long something is, draw with a pencil the part of the foot that we should use to make a unit.

2. Here is a new unit for measuring length. Fold your paper strip so that is one half times as long. How can you tell that it is ½ times as long?

3. Use your new unit to measure some of the same distances that we measured before. What do you notice?

4. How tall is this building?



Making Sense of Our Travels

Unit 4

Mathematical Concepts

- Fractions represent partial units.
- 1/b represents 1 copy of a unit partitioned into b congruent parts.
- a/b represents a copies of a unit partitioned into b congruent parts.
- a/b can be interpreted as traveling from zero to the location a/b.
- a/b can also be interpreted as iterating a length of 1/b, a times.
- 2-splits of units can be composed (e.g., ½ of 1u results in ½ u and ½ of ½ u results in 1/4u).
- Equivalent fractions mark the same distance traveled or location if the unit is the same. Hence, ½ unit = 2/4 unit = 4/8 unit.
- Relational thinking is assisted by the language of "times as long."
- Understanding the number-line.

Unit Overview

This unit revisits concepts of measurement, but in this lesson the measurement units are not just tied to feet. Students are introduced to fractions as partial-units. The goal for students is to measure distances on a street and later transferred to a number line. The long side of the rectangle is what is used to measure length, but its 2D structure helps students better visualize the results of folds. Classroom discussion focuses on splitting units, so that lengths that are not multiples of whole numbers can be measured. Fractions, a/b, are quantities representing *a* copies of *b* congruent partitions of the unit. The symbolization a/b corresponds to partitioning the unit into *b* congruent partitions by folding, and then traveling (walking) *a* of these partitions, starting at the zero. Hence, ¼ unit represents traveling from the origin, 0, to the end of the first of 4 equal partitions of the unit. Similarly, 3/4 unit represents traveling from the origin to the end of the 3^{rd} of 4 congruent partitions, and 5/4 unit represents traveling from the origin to the end of 5 of these congruent partitions, each of which is ¼ unit long. Traveling is complemented by iterating; a/b unit is *a* iterations of $1/b^{th}$ partition of the unit. Hence, 2/4 unit is 2 iterations of $¼^{th}$ unit.

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- Main Street 8
- Partitioning Problems 9
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Materials & Preparation

Read

- ☐ **Unit 4**
  Start by reading the unit to learn the content and become familiar with the activities.
- ☐ **Mathematical Background and Sample Student Thinking**
  Reread the Mathematical Background and Student Thinking boxes to anticipate the kinds of ideas and discussions you will likely see during instruction.
- ☐ **Measurement Construct Map**
  Read the construct map and look at the multimedia map to help you recognize the mathematical elements in student thinking, and to order these elements in terms of their level of sophistication.

Gather

- ☐ Student math journals
- ☐ Teacher journal for note-taking
- ☐ Paper unit strips (enough for each student to have 2 – 3 strips and at least 10 extras to use for classroom demonstration) These strips need to be the same length as the paper strips used in the town street below.
- ☐ Tape
- ☐ Town Main Street map with unit strips with landmarks to describe location. Gather different clip art (or student drawings) with different items (houses, shops, school, playgrounds, trees, flowers etc). It might be best to laminate the town and the pictures so that they can be used repeatedly to explore the ideas of distances from a point of origin and making sense of units and partitions and to record students thinking. Leave space to add other items for
- ☐ Laminated sentence strips that can be used for a number line or butcher paper (still best to laminate so you can write on and use it repeatedly with different conversations). This is an extension that will relate directly to 3rd grade CCSS Fractions. Have extra sentence strips for folding units and iteration available. Three or four sentence strips should be long enough. You may want to tape it to butcher paper so that you do not have to start over in labeling the number line if you lesson goes over a class period.
- ☐ Some toy or item to use in traveling
- ☐ Writing assessment task instructions (Appendix)

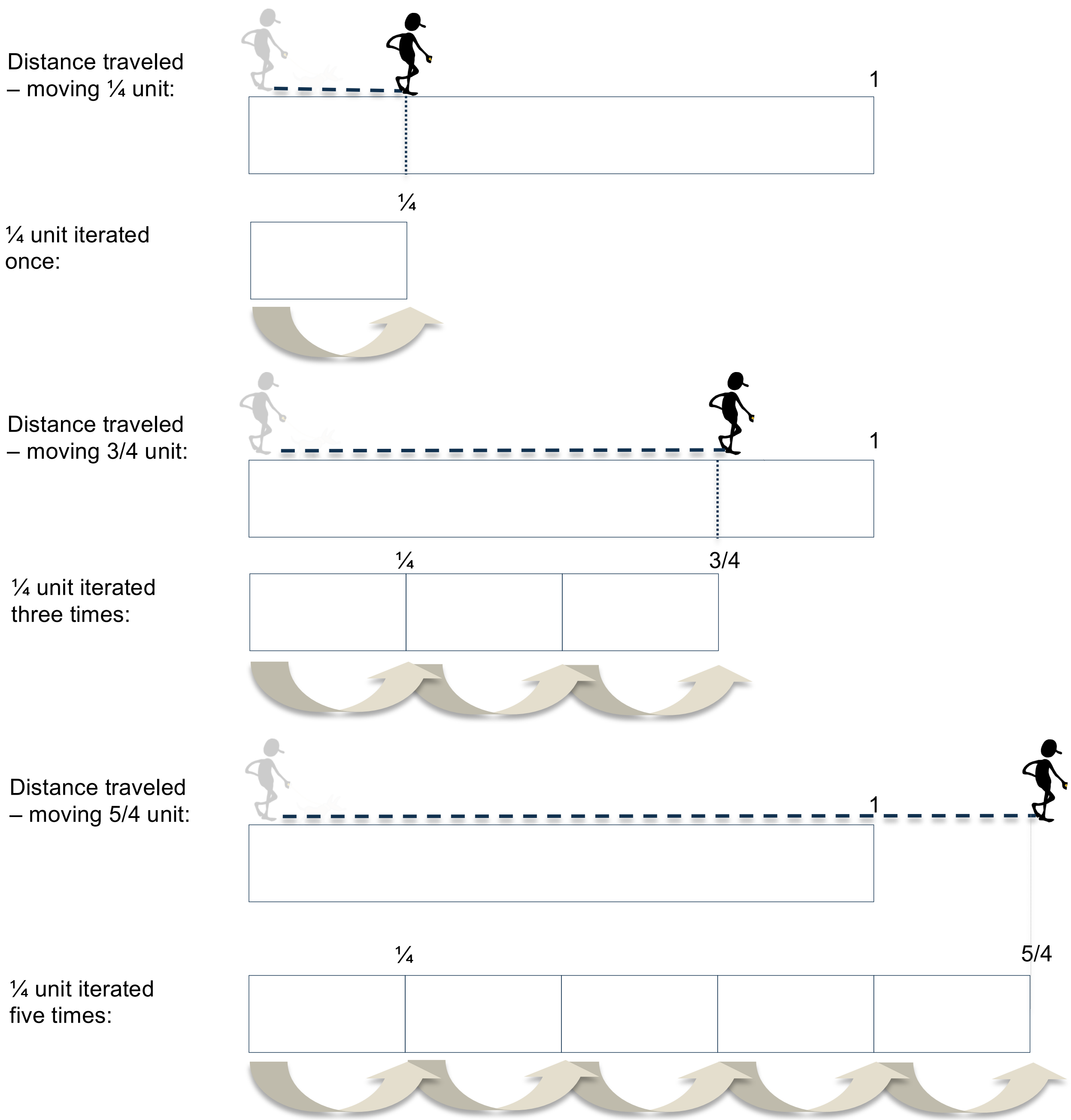
Materials & Preparation

Prepare

- ☐ Paper strips for each student for testing ideas
- ☐ Make Main Street map
- ☐ Laminate sentence strips for number line

Mathematical Background

There are two metaphors used to help students understand fractions. One relies on unit iteration where, for example, 3/4 means 3 iterations of ¼ unit. This sense of fraction relies on a copy of a partitioned unit, as in 3 copies (iterations) of ¼ unit results in 3/4 unit. The second metaphor is distance traveled where, for example, 3/4 means starting at 0 and moving 3/4 unit. Both of these metaphors, one static and the other dynamic, help students form images of fractions that will later help them locate fractions on the number-line, because the number-line is an idealized ruler.



Introducing the Unit

Each student will get 3 paper strip units; the goal is to create parts of units so that the measurement will be more accurate. Students discuss possible reasons for partitioning. What does partitioning help accomplish?

Whole Group

1. **Introduce paper strip units.**

   a. Facilitate a discussion about issues students had with their footstrip units focusing on issues with accuracy and problem solving around measurements that were partial unit lengths.

   b. Ground the need for partial units by reminding students that when they used the footstrip unit to measure, some students mentioned that some of the things you were measuring did not come to the end of a unit. (Have several students' footstrip unit handy for demonstration purposes.) Ask:

   Q: Who can remind us of that conversation?
   Q: What problems did you have?
   Q: What did you do when the object you were measuring did not end where a unit ended?
   Q: What did you have to think about?

2. **Demonstrate Partitioning.**

   a. Begin by asking: How could I split my unit so that I could measure exactly something that was one-half times as long as this unit? Most often, students will suggest folding it. If they do not, proceed to demonstrate.

   b. Ask: How should I fold my unit to make 2 parts, each exactly the same length? Then, demonstrate splitting the unit by folding into 2 congruent pieces. Ask:

   Q: How do I split a unit to make sure the parts are the same (identical, congruent) length?
   Q: What do I call each part?
   [One-half <unit name>, e.g., one-half Ricardo; **be sure to mention the unit!**]

Q: How do you know it's really a half?
Q: How far have I traveled along my unit? (Demonstrate moving half way across your unit.)
Q: What do I call the length from here *(start)* to here *(midpoint)*? [Move fingers along the paper strip. Students will likely say "one-half."]
Q: One half of what?
[One half of <unit name>]

3. Build Relational Thinking

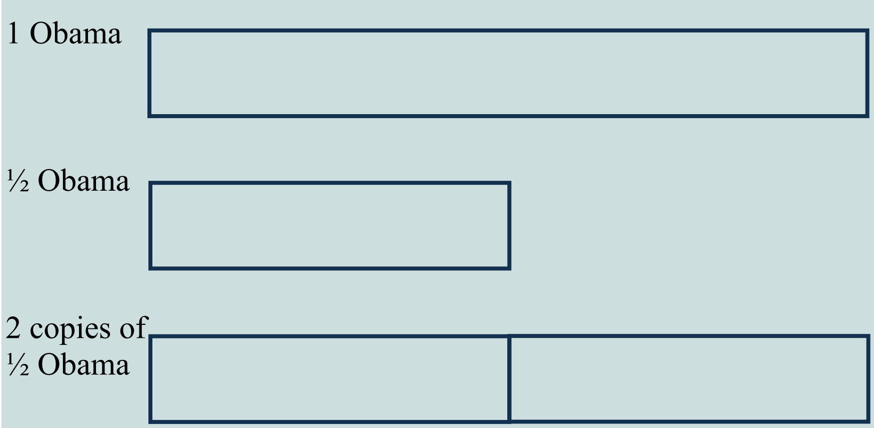
a. Ask:

Q: How many times would you have to copy ½ <unit name> to make 1 <unit name>? (2 times)
Q: 1 <unit name> is how many times as long as one-half <unit name> (2 x)

Student Thinking

The construction "times as long" may seem awkward, but it is a gateway to building reasoning about relations. This is often called algebraic reasoning and multiplicative reasoning. So, please stick with it. If your class seems to be grasping these ideas quickly, you might ask: How many times would I have to copy ½ <unit name> to make 2 <unit name>? *Keep the unit and the copy of the unit separated*, so that students do not think we need only make 1 copy of ½ unit to make 1.

For example:



4. **Write:**

   a. How many one-half <unit name> are in 1 <unit name>?

   b. 1 <unit name> is ? times as long as one-half <unit name>

      Alternate phrasing: One Nina is how many times as long as one-half Nina?"

   c. How many one-half <unit name> are there in 2 <unit name>?

      2 <unit name> is ? times as long as one-half <unit name>

      Alternate phrasing: How many times do you have to copy ½ Nina to make a length that is 2 Nina units long?

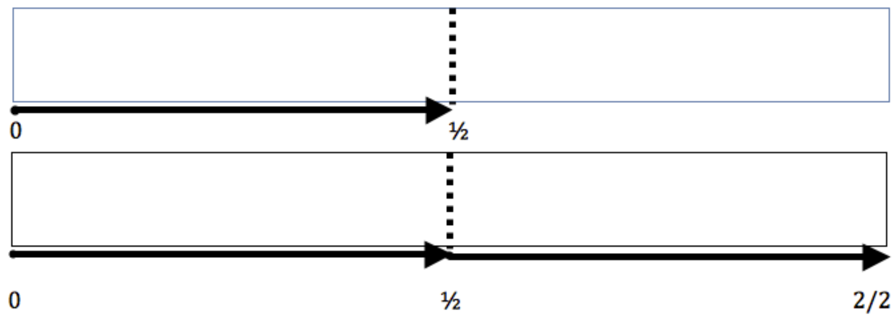
   d. Have the students use of iteration to prove the relation (move the half-unit 2 or 4 times) working with a partner. Then have someone come up to the front of the group and prove it for the class. Discuss.

5. **Introduce students to writing fractions: how might we write one-half with numbers?**

   a. Here is how we write so that other people know what we mean: ½. The bottom number tells us how we have *split* our unit—how many congruent parts. *Congruence means that the parts match exactly, so when we put one length on top of another, the distance is exactly the same.* The top number tells us how many copies of these parts we have. So, if we have ½, it means that we have 1 copy of the unit that we have split into two congruent parts, so it should take 2 of them to make one unit.

   b. Review the previous demonstration by iterating ½ twice to make 1 unit. Then emphasize travel: So, when we travel ½, we start at zero and travel all the way to the end of the first part (see illustration on the following page). If we keep traveling, and travel all the way to the end of the second part, we write 2/2. This means that we have traveled 2 of the ½ units.

Introducing the Unit
Main Street
Partitioning Problems
What Have We Learned?



c. Ask:

Q: Thinking about this ½ unit, what does 2/2 unit mean? *(Notate their ideas as they describe their thinking. At this point, introduce mathematical notation of 1/2 (the bottom number or denominator is the number of splits and the top number or numerator is the number time the unit has been iterated). Begin to press on mathematical relationships and communication through more formal notation. For example: ½ + ½ = 2 x ½*

The goal is to notate what they say. If you have not yet introduced the multiplication symbol, it does not hurt to tell them that mathematicians also use this symbol 2 x ½ to represent the number of times a group is repeated. Make sure that you connect it to repeated adding. The goal is to make connections. Some students will get the notation and start using it. Students who are not ready will not use it. That is okay.

Q: What about 3/2 units? ½ + ½ + ½ = 3 x ½

Q: Working with a partner, travel from the beginning (0) to ½ unit. Now travel another ½ unit. Then travel another ½ unit. How far have you traveled? (½ + ½) + ½ = (2 x ½) + (1 x ½) = 3 x ½

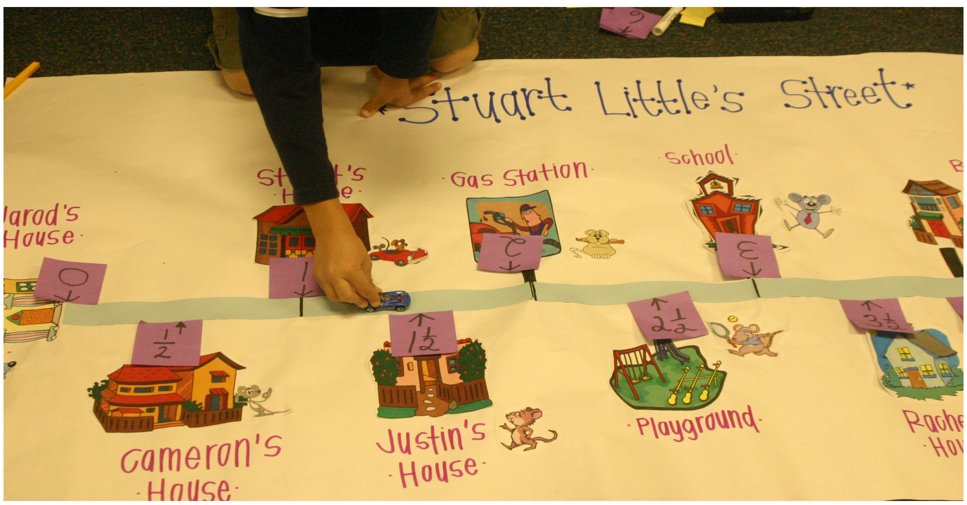
Q: What about if you traveled another ½ unit? How far would that be altogether?

Note. Establish a relation between *(for example)* a measure of 4 units (also written as 4/1) as 4 iterations of a single unit and 3/2 as 3 iterations or copies of ½ unit. ½ means 1 copy of ½, 2/2 means 2 copies of ½, 3/2 means 3 copies of ½, 4/2 means 4 copies of ½. We want children to get comfortable thinking about measurement as iteration of a unit, but at the same time, we want children to start to think about "times as long" to consider relations between splits of units and the original unit. Language such as 3/2 unit is 3 times as long as ½ unit, and 3/2 unit is 1 ½ times as long as 1 unit helps students think about these relations.

Introducing the Unit
Main Street
Partitioning Problems
What Have We Learned?

Main Street – Using Context

Whole Group



6. Explore these ideas using the road map with (Jacobsville, Mousetown or whatever you choose to name your Main Street). Your class will use Main Street as a way to further explore the big ideas of the number line and measurement (zero point, distance from the point of origin, need for partitions (halves and fourths), comparing the relationships between whole units, half units and fourth units, iterating partitions, and identifying congruent distances in context.

You can begin by having them begin with one <name your unit> and establish the landmark at the end the street is at zero. The town starts at the (landmark). Then using a critter or your hand moving down the street, have them identify where one <name your unit> is, two <name your unit> is and mark them on the map and label it.

Extend your discussions by identifying halves on Main Street. Have the students describe different distances using half a unit.

Q: If I start (zero point landmark) and travel by half units to the landmark 2 units away (describe the landmark – do not quantify how far it is away), how many half units will you have traveled? How do you know? How else can we describe how far you have traveled? *Record the notation on the board they describe: "So you are saying your traveled four half units so ½ +½+ ½ + ½ = (or) 4 x ½.*

Q: How else could you describe the distance traveled? Are there other names that you could describe the units? Record these equivalences on the board as you work on Main Street. "*So you are saying that you traveled 2 whole units or 2/1. So are ½ +½+ ½ + ½ = 4 x ½ = 2/1?"*

Note: Repeat this discussion by having the students travel to different locations on the main street. (Continue to use mathematical notation to describe travels.) Make sure to include distances that:

- Are not always traveling from zero point but from other locations (points of origin).
- Are not whole units (i.e. 3 ½ units)
- Are not whole units and travel to another point that is not a whole unit (a half unit location to a different half unit location)
- Travel backwards up or down Main Street (does direction matter). If I am here and travel back, how far have I traveled?
- Travels from a half unit to a whole unit

Partitioning Problems

Introducing the Unit
Main Street
Partitioning Problems
What Have We Learned?

After students have played with exploring concepts on Main Street, move to using a number line to continue to explore the same ideas. You can have students make a number-line on their directly on desks by having them tape 3 – 4 paper strips on butcher paper. Also have one up on the board (using sentence strip units so partitions can be seen easily across the room).

Students work in pairs to solve two different partitioning problems involving one half and then one quarter.

Partner

7. Students work in pairs to solve problems using the number lines on their desk.

a. Present problem 1: How could I make ¼? Post the following questions:

Q: How could we fold so the result is ¼ of 1 unit?
Q: What would we need to think about if we wanted to find a length that is ¼ of 1 unit?
Q: How would we know it is ¼?

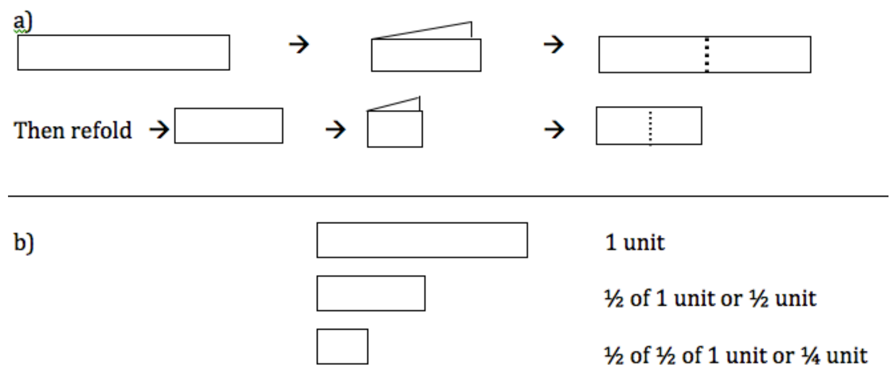
b. Students should discuss the questions with a partner and then use paper strips to test their ideas. Tell students they should be able to explain and demonstrate their thinking so they can convince their classmates they have found ¼ of 1 unit. Have students record their thinking into their math journal.

c. Observe as partners work, and support thinking about strategies. Take notes about student difficulties. Try to get each pair of students to see how they are thinking about fractional parts and labeling those parts, without giving direct instructions.

Note. This problem may or may not be challenging because it asks students to compose splits of a unit: ½ of ½ (of 1 unit). If the students discovered this relationship in earlier units, it will be a matter of re-discussing it. Let students work in pairs to answer the first problem, then return to the whole group. Compare solution strategies (see discussion questions on the next page). One common strategy is to fold the paper in half twice. Be sure to allow children to perform this folding. After the first fold, ask children to name the part of the unit. Review the notion that the unit has been split into two congruent pieces. Then, refold the unit, and ask children to fold the ½ by ½ again. Is it often helpful to tape these successive actions on the unit strip to the board, so that students can see 1 unit (it is always important that students see 1 unit), ½ unit, and ¼ unit. Ask student to unfold their strips and count the number of partitions. Ask how they might say and write the distance traveled between the beginning and end of one of the 4 partitions (¼ or one-fourth). Some students may not be challenged by this problem or may finish it quickly. If so, ask them to try to create 1/8 unit. Depending on the grade level, you can decide if

Introducing the Unit
Main Street
Partitioning Problems
What Have We Learned?

your students are ready to explore eighths or not. If you do, follow the same discussions as you did with halves and fourths.



Whole Group

2. Ask students to display their results for finding ¼ unit.

a. Post the strips that students offer, and ask:

Q: Why do you think this shows ¼ of 1 unit?
Q: How far is it from here (begin at 0) to here (first fold line should be ¼)?
Q: How do you know for sure?
Q: How could we test it?
Q: How did you create the ¼ unit?
Q: How many copies of ¼ unit are needed to make 1 unit?
Q: How can you tell? (Connect iterating ¼ unit 4 times with 4 copies of ¼ unit.)

Note. Students may have used two consecutive vertical folds to find ¼, or they may have made 1 vertical split and 1 horizontal split. If this happens, be prepared to compare and discuss their 2 different methods of folding and the difference in results. Make sure to ask which ¼ unit would best serve measuring *distance traveled*. The vertical and horizontal split will be a good way of measuring area. Students may also present a unit that has been split twice, but they may open it up. This will call for a discussion about how much of the unit is showing and what we could call it (4/4 or 1 unit split into 4 equal lengths). Then compare it to the unit folded to show ¼. Ask: How would we write ¼? Why would we write ¼ that way? What does each part of the symbol mean? Be sure to symbolize this as ½ of ½ of 1 <unit name> (as well as ¼) to emphasize that the symbolism captures what is essential about the activity—even though the result is a different length for differing personal units, the result is the same: 1 is now measured in 4 of these new sub-units. Emphasize again that the part-of-the-unit traveled is one-fourth of the length of the unit and that the unit is four times as long as the part-unit. If students are finding this well within their grasp, ask them to compare 2/4 unit to 1 unit: 1 unit is 2 times as long as 2/4 unit.

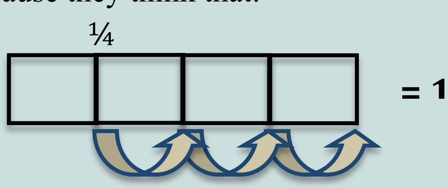
Partner

3. Students work in pairs to solve problem 2 to build relational thinking. Ask:

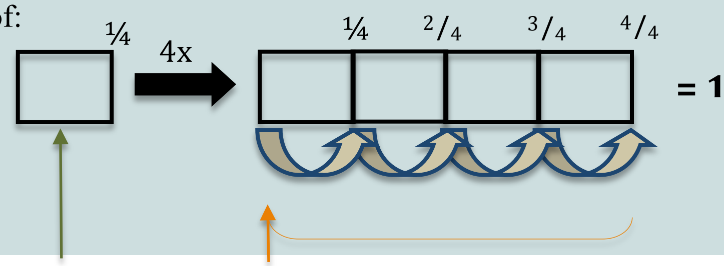
Q: How many copies of ¼ <unit name> do we need to make 1 <unit name>?
Q: 4 of ¼ <unit name> is 1 <unit name>.
Q: 1 <unit name> is ______ times as long as ¼ <unit name>.

Student Thinking

Another way to emphasize the reciprocal relationship between the part-unit and the original unit-length is to ask students to consider how they might recreate the unit-length using the partitioned unit. For example, 1 unit can be recreated by iterating the ½ unit twice; 1 unit can also be recreated by iterating the ¼ unit 4 times. It is important that children see that there are 4 iterations of ¼ to recreate 1 unit. Some may say that it only takes 3 iterations, because they think that:



instead of:



Keep **the unit** and **its copies** literally separate.

We are aiming for the time when students will anticipate that 4 iterations of ¼u *must* result in 1u, 8 iterations of 1/8u *must* result in 1u, and more generally b iterations of 1/b u must result in 1 u. You can help students make this transition by asking them to consider what changes and what stays the same when they re-create the unit starting with one-half, one-fourth, one-eighth of the unit.

Whole Group

Introducing the Unit
Main Street
Partitioning Problems
What Have We Learned?

4. To build equivalence, post the strips that students offer.

a. Ask:

Q: Travel ¼ unit. Now travel another ¼ unit. How far have you traveled?
Q: Travel ½ unit. How far have you traveled?
Q: How is 2/4 unit the same as ½ unit? How is it different?

b. Explain that when we travel the same distance from the beginning (we land at the same place), we call these distances equivalent. So, 1 copy of the unit split into 2 partitions (½) means the same thing as 2 copies of the unit split into 4 partitions (2/4).

Note. For the same unit lengths, equivalence means the same distance traveled, although the split-units might be different. That is, ½ <unit name> vs. traveling 2 of ¼ <unit name>. Clearly, ½ and ¼ are not identical because they represent different partitions of the unit. This lack of identity is confusing to some children, who think that "equal" is a synonym for "exactly the same." This is another opportunity to build relational thinking.

What Have We Learned?

Individual

1. **Introduce student writing assessment task.**

   a. Ask students to spend approximately 15 minutes writing a reflection in their math journals telling how their thinking changed from measuring with foot-strip units and traveling on Main Street or the number line.

   b. Tell students you will choose several journal entries to be read to the whole group during the next measurement lesson. Allow students to write unprompted, as this is a nice opportunity to see the topic on which the students focus.

   c. Display the instructions on chart paper (or use the page provided in the Appendix). Instructions are as follows:

   Write complete sentences telling what you know about measurement. After you finish your reflection, go back and read your previous measurement entries, then make any necessary additions to your reflection. Some questions to consider are:

   Q: How is your thinking the same?
   Q: How has your thinking about measurement changed from your previous journal entries?
   Q: What questions do you still have about measurement?
   Q: Include in your reflection: Where do you start measuring? What do you call it?

   d. Read through the entries before the next class and choose 3 to share. Write a summary of your three focus students, using the following questions:

   Q: How are your students thinking about fractional lengths?
   Q: What surprised you?
   Q: What, if anything, did they find difficult?
   Q: Do students understand the advantages of using a standard unit

Written Response

a. Ask students to spend 15-20 minutes writing a response to the question below in their math journal.

   Draw a picture of Main Street or a number line. Label it to show what you understand about what is important about measurement. Explain your thinking.

b. Read each written assessment and analyze student understanding of measurement in relation to the progress map. Are students making progress? Take notes on the understanding and misunderstandings of your focus students. Use this summary to prepare for Unit 5 demonstration and discussion.

Connections to Conventions

Mathematical Concepts

- A linear measure unit can be iterated to create a measuring tool that is a unit of units.
- Measurement tools utilize equivalent units and partitions.
- Standard units facilitate comparisons.
- Constructing and analyzing conventional units.
- Making connections between the concepts of linear measurement, non-standard units and conventional system (inches, feet and yards).
- Using and reading a ruler (from both zero point and point of origin).
- Extending understanding and mathematically describing the results of measuring with a measuring tool.

Unit Overview

Starting with units that are 2-inches long and a paperstrip that is 12 inches long, the students will make a measuring tool by iterating their unit to create a ruler and label it. The class will analyze a set of measuring tools (selected purposefully by the teacher) to examine issues related to understanding and labeling a measuring tool.

Next students will use units that are 1 ½-inches long and a paper strip that is 18 inches long, the students will make a measuring tool by iterating their unit 12 times to create unit of units and label it. Then they will measure select items that force the students to partition their units for more accuracy and to continue to press on the ideas of fractional units, point of origin, iteration and the use of ruler.

Finally, students will make comparisons between their measuring tool and the U.S. Customary Ruler. How are they alike and different? What big ideas is this ruler based on?

Unit

5

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Materials & Preparation

Read

- ☐ **Unit 5**
  Start by reading the unit to learn the content and become familiar with the activities.
- ☐ **Sample Student Thinking**
  Reread the Student Thinking boxes to anticipate the kinds of ideas and discussions you will likely see during instruction.

Gather

- ☐ Student math journals
- ☐ Teacher journal for note-taking
- ☐ Paper strips that 12 inches long (1 per student) and paper strips that are 2 inches long (2 per student)
- ☐ Paper strips that are 18 inches long (1 per student) and paper strips that are 1 ½-inches long (two per student)
- ☐ Extra paperstrip units for the teacher to use as a demonstration
- ☐ 12 inch rulers with markings to fourths only
- ☐ Chart paper, markers

Prepare

- ☐ **Prepare Unit Lengths -** Prepare paper strips that are 1 inch x 12 inches for the development of the unit of units (1 per student) and 2 inches long for the unit (2 per student).
- ☐ **Prepare Unit Lengths -** Prepare paper strips that are 1 inch x 18 inches for the development of the unit of units (1 per student) and 1½ inches long for the unit (2 per student).
- ☐ Identify items that students can measure that can be measured accurately to the fourths. Identify items that can be measured to fourths with the following properties or make items (or paper stips) that can be measured using the student rulers.
  - o Less than a 18-inches long that results in a measure that is a whole unit
  - o Less than a 18-inches long that results in a measure that results in half and fourth partitions.

Instruction

Introducing the Unit

Introducing the Unit
Exploring Compositions & Conventions
Making a 12 Unit Ruler
Comparison to U.S Customary

Students participate in a whole-group discussion about important lessons learned, before considering problems in small groups or partners.

Whole Group

1. **Introduce the task: summarizing journal reflections, introducing standard unit, 2-splits of the unit.**

   a. Ask students to summarize their journal reflections or choose the journals of 3 different students for a discussion of "lessons learned" by comparing scales in the previous unit.

   b. Revisit the importance of the origin of measurement as zero and the rationale for how units are labeled.

   c. We have been using different units to measure lengths. If we all agreed to use the same unit of measure, would that make easier? (Standard units are agreed upon to make comparisons of lengths of different objects easier.)

   d. Why do we split units? What does that help us do (increase accuracy of measurements)?

   e. Why do we have measuring tools with units of units (for measuring larger objects or distances)?

Note: The goal here is to review the key ideas that the students have explored in the past 4 units to help them ultimately connect to standard conventional units.

- Zero point
- Partitions are needed for accuracy
- Iterations of units are needed when measuring larger objects or units

Exploring Compositions & Conventions

Individual/Student Teams

1. Students make a 6-unit measuring tool.

a. Introduce the <name of unit> unit (2 inches long) and the <name of unit> unit (12 inches long). The students will make a measuring tool by using their <Jacob> units and iterating it to make one <Anthony> unit (ruler or measuring tool) and labeling it.

b. Give each student a 2 <name of unit> units (2 inch strips) and one <name of unit> unit (12-inch strip). Then have the students make a measuring tool by iterating the <Jacob> unit across the paper strip (one <Anthony> unit) and label their rulers. (At this point, let them partition and label as much or as little as they choose. This will be part of the discussion later.)

2. Students compare different 6-unit tape measures.

c. Select measuring tools that exemplify the ideas from ones described below in the Student Thinking box to share or is likely to reveal students' conceptions/misconceptions of the measuring tool. (This is going to be heavily dependent on what your students do and what ideas they still need to make sense of.) If no one makes the measuring tools that have qualities you need, make some so that you have examples that can bring out ideas that do not come naturally from theirs. Students are not likely to make a tool like D. It is a good one to share (but it is important for students to think this was created in a different class as opposed to offering it as a standard you want them to consider). Students compare the measuring tools, identify which would be problematic, and which are good examples of a tool and why. Have something that could be measured (at least to the half unit and ¾ unit). Discuss.

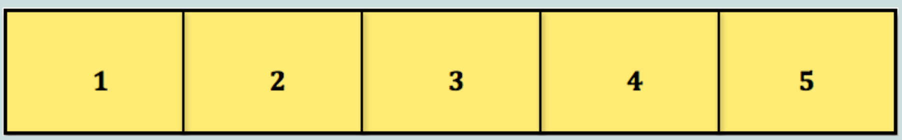
Then have them choose which they believe is best tool, explain their thinking and write about why they think in their journal. If students make other errors that seem to be typical, include those as well in the set for discussion.

Introducing the Unit
Exploring Compositions & Conventions
Making a 12 Unit Ruler
Comparison to U.S Customary

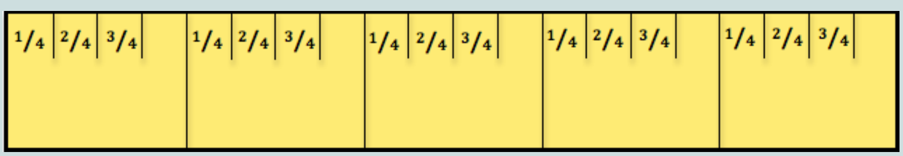
Student Thinking

The tapes are designed to support students' consideration of the role and functions of numeric symbols in systems of measure and make connections from the act of measuring and how it connections with conventional tools.

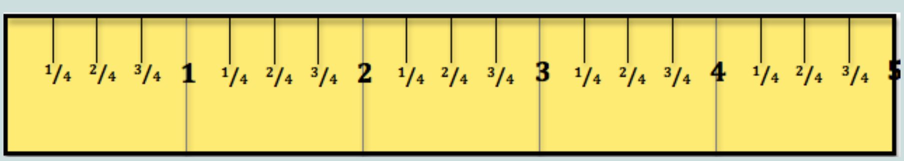
Tape A has 5 equal partitions but labels each partition in the interior. This makes it difficult to know what is meant by "1" or "2." Moreover, the units are not split, making measurements of lengths difficult if the object does not correspond exactly to whole numbers of units.



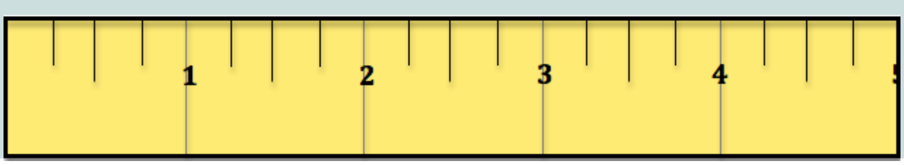
Tape B has 5 equal partitions, and each partition is further split into fourths. But the labels are ambiguous, so it is difficult to know the length of "3/4" or any other split-unit. The whole number units are not labeled.



Tape C labels whole units and split units so that the label corresponds to a distance traveled.

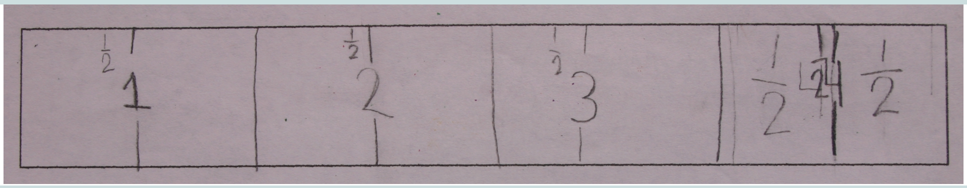


Tape D labels whole units only, splits the units, and signifies ½ and ¼ partitions differently. It relies on user knowledge of the meaning of the split unit. This tape is closest to those that we use conventionally.

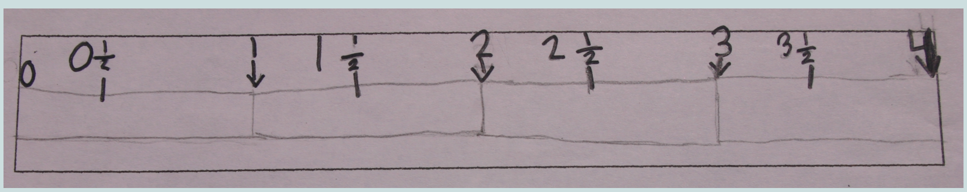


Introducing the Unit
Exploring Compositions & Conventions
Making a 12 Unit Ruler
Comparison to U.S Customary

Tape E labels the unit in as a count without regard to zero point but labels half units using zero point (distance from the beginning of the unit). This causes difficulty as the ruler mixes conventions (area and linear model) and makes interpretation and use of the ruler difficult.



Tape F labels the ruler thinking about the distance from zero but forgets about the importance that halves are equi-distance measures.



Whole Group

3. **Students identify the measuring tools they prefer and which tend to be problematic.** Have the students identify which rulers they think work and which do not. They should justify their thinking about what id productive and what is not. This will provide some formative assessment in terms of what students understand about measure and how that thinking relates to a measuring tool.

   a. Questions to promote discussion include:

      Q: Which is best for finding a distance traveled (indicate with hands) that measures ¾ unit? Why?

Tape A, with the interior label (and no partitions marked), makes this distance ambiguous. Tape E is labeled only to ½. Tape E has issues of the half marked as a distance from zero but the unit labeled in the space (area model). They are also not labeled to fourths.

Q: What is problematic about tape F? *Tape F is labeled with zero point in mind. The units are equal but the partitions are not equal.*

Q: If you used Tape E & F to measure, what might be confusing? What would you do with these measuring tools to make it so that they worked better?

Q: Which is best for finding a distance traveled (a length that is as long as) of 1 ¾ units? Why?
Either the explicitly labeled tape measure (C) or the implicitly labeled tape measure (D) will do.

Q: What do you think the people who made tape D had in mind? What do you think these people assumed you already know about measurement?
This ruler is closest to standard practice—but it relies on the users knowing the "rules."

4. After the class has had their conversations and stated their views, have two items available that students can measure that results in a measure that is related to first a half and then a fourth. This will help to see if they can prove how the rulers of their choice are useful for measuring and which are not.

Making a 12 Unit Ruler

Introducing the Unit
Exploring Compositions & Conventions
Making a 12 Unit Ruler
Comparison to U.S Customary

Individual/Student Teams

5. Introduce the <unit name> unit (1 ½ inches long) and the <unit name> unit (18 inches long). Have the students make a measuring tool by using the Alyssa unit (1 ½ inches long) and Clint unit (18 inches long) units. We will then have them use them to measure selected items.

d. Today, we are all going to use this unit (one Alyssa unit) to make a measuring tool (one Clint unit). Then we will use them to measure items in the class.

e. Give each student 2 Alyssa units (1 ½ inch strips) and one Clint unit (18-inch strip). Then have the students make a measuring tool by iterating their Alyssa unit across the Clint paper strip) and label their measuring tool. (At this point, let them partition and label as much or as little as they choose. This will be part of the discussion later. Hopefully after the previous conversation, most students' rulers will be done correctly but if not, this is one more time to discuss and consider the important ideas behind the construction of a measuring tool. This will be both a formative assessment (how much did they make sense of the ruler and what are they still struggling with) and the foundation for the classroom discussion.

f. Have the students measure the set of items in the room and record their answers.

Whole Group

Student Thinking

As students construct their measuring tool, probe their understanding of how to label the partial units.

What do students call each partition? We want to ensure that students understand that each partition is called 1/b, where b corresponds to the number of congruent partitions of the unit.

Where do they write the numeric label? This is often particularly revealing. Students who do not label the unit at its endpoint may not have a firm grasp on the measure as a distance traveled.

After students have constructed their measuring tool, be sure they try to use them to measure a length that is *less than* the length of the unit, and a length that is *greater than* 12 of the units.

3. Use the students' measures and thinking to press on further on the big ideas about measuring tools, understanding partial units and zero point and conventions for labeling measuring tools.

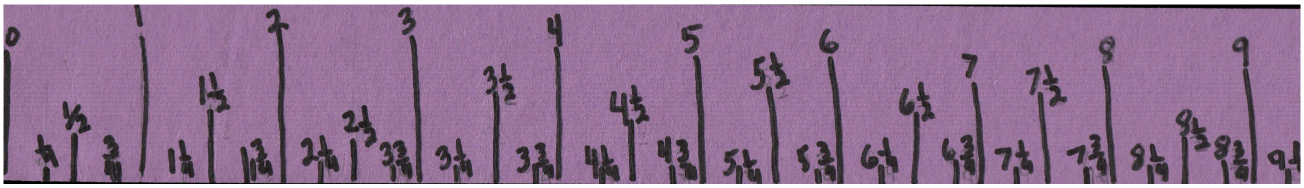
Introducing the Unit
Exploring Compositions & Conventions
Making a 12 Unit Ruler
Comparison to U.S Customary

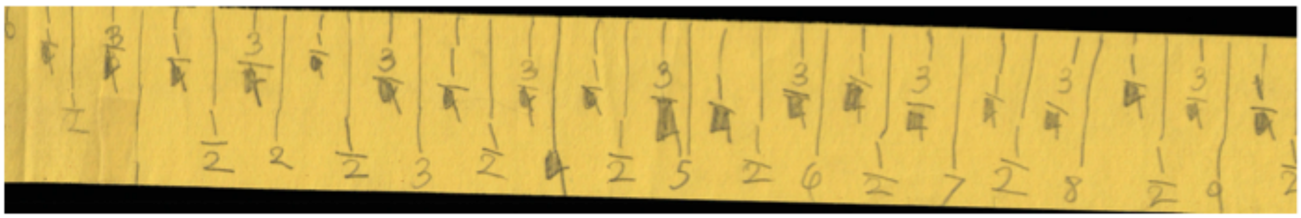
a. Conduct a whole group conversation to explore the issues related to measuring tools and measuring. Choose student responses to the questions addressed individually and with partners.

1) Begin by having the students share measures that resulted in whole Alyssa units and less than a Clint unit. Look for:
    a) How did the students label their rulers? Did they label the whole units, if so where did they label it?
    b) How did they describe their measure? Were their measures influenced by students understanding of the big ideas of measure? If not, what big ideas are they struggling with? What is your evidence?
    c) Does their thinking reflect understanding about zero point?
    d) Ask the students to describe and justify their thinking?
        i. Why did they label it at the end of the unit or the middle of the unit? Does it matter? Why does it matter?
        ii. Have them compare and contrast the different measures for the same items. Ask them why there are different measures? Are these measures equal to each other or not? Ask them if they can justify their thinking.
2) Next have them look at measures that less than a Clint unit but results in partitions of the Alyssa unit.
    a. Again, how did the students label their rulers? Did they just label the whole Alyssa or did they partition them? Did they label the partial units? If so, how did they label? Did they label the partitions reflect understanding of zero point (i.e. ¾ is ¾ from zero or did they label all partitions as ¼, ¼ , ¼ … regardless of the distance from zero).
    b. This is another place for students to talk about zero point and distance from zero. It will also give you a chance to talk about equivalence if students quantify their measures differently.
    c. Place items on the ruler but do not start at zero (different point of origin).

Did they label their ruler continuously 1 ½, 1 ¾ , 2 ¼ , 2 ½ , 2 ¾ or by non-continuous partitions ¼ , ½ , ¾. Did they let the lines show the different partitions? Which is the easiest to read? What if you do not start measuring at zero? Which tool is easiest to measure with?

Introducing the Unit
Exploring Compositions & Conventions
Making a 12 Unit Ruler
Comparison to U.S Customary





d. Ask the students to describe and justify their thinking.
   i. Have them compare and contrast the different measures for the same items. Ask them why there are different measures? Are these measures equal to each other or not? Ask them if they can justify their thinking.

Comparisons to U.S. Customary

Introducing the Unit
Exploring Compositions & Conventions
Making a 12 Unit Ruler
Comparison to U.S Customary

Individual/Student Teams

6. Pass out standard U.S. rulers (ruler should only have fourths labeled) out to each of the students. Have the students examine this tool and ask them to compare and contrast this measuring tool to the ones that they invented. How are they alike or different? (same markings but enlarged)
   a. Try to pull out that both measuring tools consists of 12 units.

b. Each unit is partitioned into fourths similar to the way some of the measuring tools were in the class.
c. Some students labeled their rulers (partitions) where on this measuring tool, the partitions are marked but not labeled.

7. Explain to the class that this is the measuring tool that we use in the United States. It is called a ruler that is a foot long (about the same as our Clint unit). The ruler (we call a foot) is comprised of smaller units (similar to our Alyssa units) and are called inches .

8. If you are using a word wall, this would be a good place to write the terms inches and feet and help them to connect convention with their inventions.

   a. Suggest to students that they have both rulers on their desk as they write the reflection. Have them include diagrams if it helps the students clarify their thinking and to support the writing skills of your grade level.
   b. Give the students some paper strips or items to measure using the U.S Customary ruler. Using the students' measures, engage them in discussions about how they are using the ruler and how the use is like their student made rulers.

