Measurement Model of Division, Grade 4

UNIT

18

Mathematical Concepts

- Division $\frac{a}{b}$ of two numbers can be interpreted as re-measuring a length, aU (first measured in the unit length U), with a new unit, bU. The result, k, of the division is a scalar multiple, such that $k \times b\text{U} = a\underline{U}$. For example, a 10 inch length re-measured with a 5 inch length results in 2 (of the 5 in. length), and 2 × 5 in. = 10 in.
- $\frac{a\ \text{U}}{b\ U} = k = \frac{a}{b}$

Unit Overview

Given a length A measured as aU, the division $\frac{a}{b}$ is modeled as re-measuring that length in a new unit, bU. The quotient, $\frac{a}{b}$, is equivalent to the ratio $\frac{a\ \text{U}}{b\ \text{U}}$, which is a scalar multiple, k, such that $k \times b\text{U} = a\text{U}$. The unit begins with teacher-led modeling of division as this sense of re-measuring. Students next work with a partner to solve problems involving whole number and fractional measures of aU and of bU. Partner work includes measuring the circumference of a circle with units of radii and with units of diameter to establish the generality of the model for any real number. The unit concludes by developing an algorithm for division of fractional quantities by finding common measures (i.e., the denominators for a and b are expressed by the same splitting factor). When the denominators are common, k is the ratio of the numerators.

Contents

Read

- ☐ **Unit 18**
  Start by reading the unit to learn the content and become familiar with the activities. Try out all the investigations yourself, before teaching.

- ☐ **Mathematical Background**
  Reread the mathematical background carefully to help you think about the important mathematical ideas within the unit.

Prepare

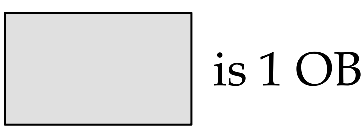
- ☐ Provide students with paper strips to represent units of measure.

Magnitude of a Length

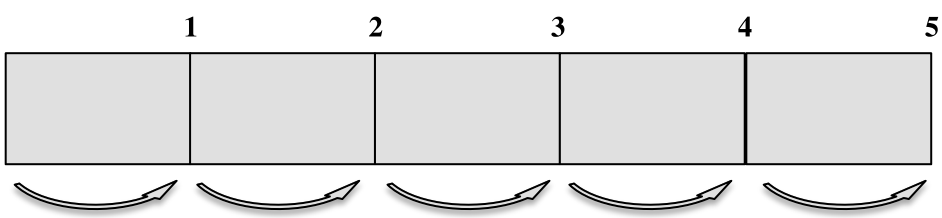
The magnitude of a length, its spatial extent, is not altered by division. The length remains constant: it neither stretches nor shrinks. Division merely re-measures the length. Scalar multiplication, in contrast, stretches or shrinks a length, unless the mutliplication is by 1.

Division as Re-Measuring a Length

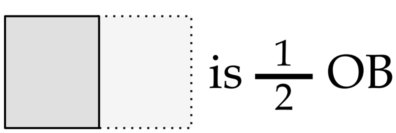
Case 1. The unit of re-measure is less than the original unit of measure. For example, for a length measured in OBama units, if the unit of re-measure is $\frac{1}{2}$ times as long as 1 *OB*, as in $\frac{5\,OB}{\frac{1}{2}\,OB}$, the result is 10. $10 \times \frac{1}{2}\,OB$ restores the original measure, 5 *OB*: 10 is a scalar multiple that acts to restore the original measure. It tells us that there are 10 ($\frac{1}{2}$ *units*) in a length that is 5 OB *units* long.



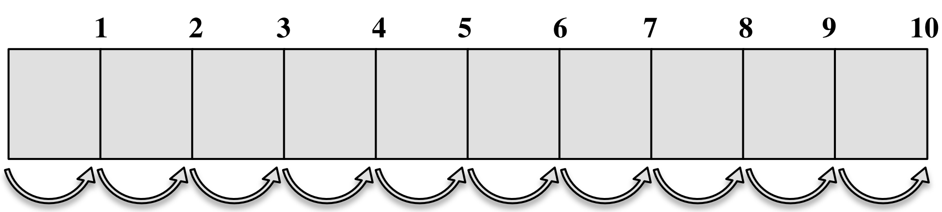
Unit of Measure



Original Length Measure is 5 OB.



Unit of Re- Measure



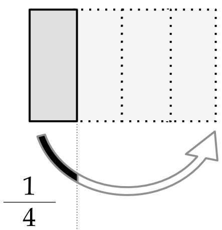
The re-measured length is 10 of the $\frac{1}{2}\,OB$.

$\frac{5\,OB}{\frac{1}{2}\,OB} = 10 = \frac{5}{\frac{1}{2}}$ and $10 \times \frac{1}{2}\,OB = 5\,OB$

A second example of the unit of re-measure as less then the unit of measure is illustrated by $\frac{\frac{1}{4}OB}{\frac{1}{2}OB}$

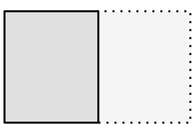
is 1 OB

Unit of Measure



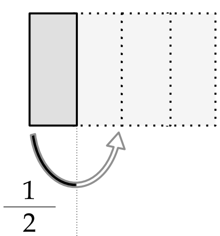
$= \frac{1}{4}$ OB

Original Length Measure is $\frac{1}{4}OB$



is $\frac{1}{2}$ OB

Unit of Re-Measure



$= \frac{1}{2}$ of $\frac{1}{2}$ OB

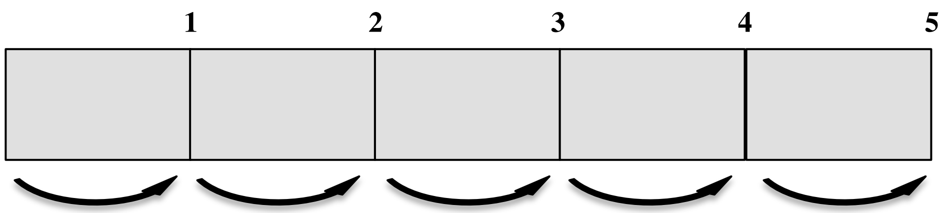
The re-measured length is $\frac{1}{2}$ of the $\frac{1}{2}$ OB

$\frac{\frac{1}{4}OB}{\frac{1}{2}OB} = \frac{1}{2} = \frac{\frac{1}{4}}{\frac{1}{2}}$ and $\frac{1}{2} \times \frac{1}{2}$ OB $= \frac{1}{4}$ OB

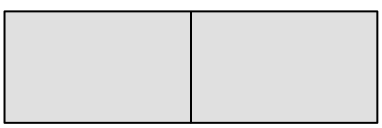
Case 2. The unit of re-measure is longer than the original unit of measure. For example, if the unit of measure is 2 times as long as 1 *OB*, as in $\frac{5\ OB}{2\ OB}$, the result is $2\frac{1}{2}$, so that $2\frac{1}{2} \times (2\ OB) = 5\ OB$, as illustrated below. It tells us that there are $2\frac{1}{2}$ of 2 *OB* $(2 \times 2\ OB\ + \frac{1}{2} \times 2\ OB)$ in 5 *OB*.

is 1 OB

Original Unit of Measure

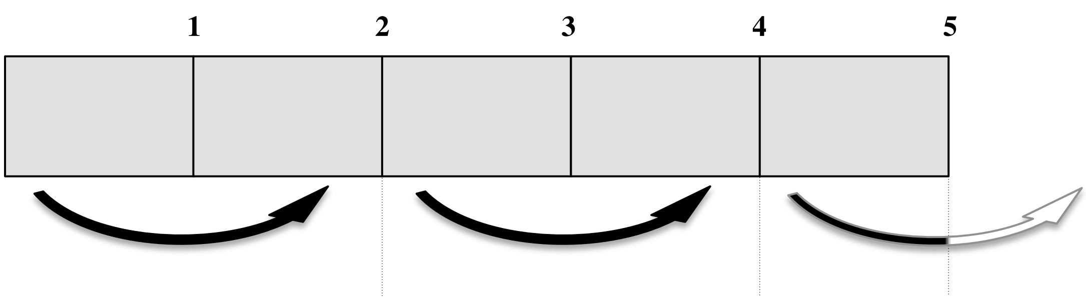


Original Length Measure is 5 OB.



is 2 *OB*

Unit of Re- Measure



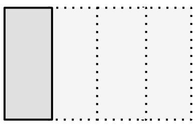
The re-measured length is $2\frac{1}{2}$ *of the 2OB units.*

$\frac{5\ OB}{2\ OB} = 2\frac{1}{2} = \frac{5}{2}$ and $\quad 2\frac{1}{2} \times 2\ OB = 5\ OB$

A second example of the unit of re-measure as longer than the unit of original measure is illustrated by $\frac{\frac{1}{4}\,OB}{2\,OB}$

is 1 OB

Original Unit of Measure

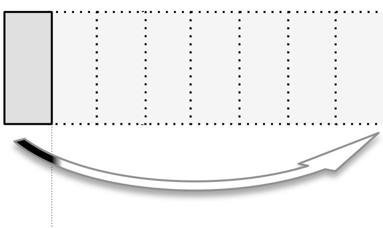


$= \frac{1}{4}$ OB

Original Length Measure is $\frac{1}{4}$ *OB*

is 2 OB

Unit of Re- Measure



$= \frac{1}{8}$ of the 2 OB

The re-measure of the length is $\frac{1}{8}$ *of the* 2 *OB*.

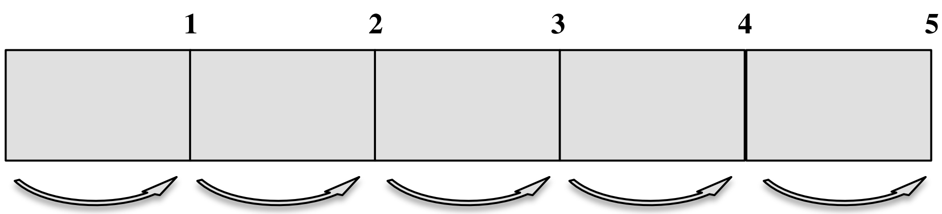
$\frac{1}{8} \times 2$ OB $= \frac{1}{4}$ OB.

Case 3. When the original and new units are congruent. Recall that the measure of length can be interpreted as the ratio of the magnitude of the length to the magnitude of the unit length. So, we can say that if a length has a measure of 5 OB, this means that $\frac{5\ OB}{1\ OB} = 5$. That is, 5 *OB* is 5 times as long as the unit length,1 *OB,* and hence, $5 \times 1\ OB = 5\ OB$.


is 1 OB

Unit of Measure

Length to be measured



$\frac{5\ OB}{1\ OB} = 5$ and $5 \times 1\ OB = 5\ OB$

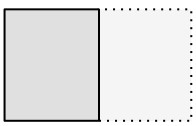
Mathematical Background

Further Examples.

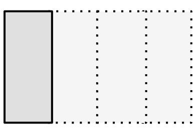
If a length with measure $\frac{1}{2}$ *OB* is re-measured using $\frac{1}{4}$ *OB*, as in $\frac{\frac{1}{2}\ OB}{\frac{1}{4}\ OB}$, the result is 2, because $2 \times \frac{1}{4}$ *OB* = $\frac{1}{2}$ *OB*. The measure of the magnitude of the length originally expressed as $\frac{1}{2}$ *OB* is 2 of $\frac{1}{4}$ *OB*.

is 1 OB

Original Unit of Measure

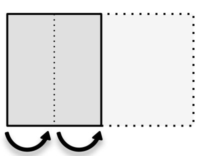


Original measured length is $\frac{1}{2}$ OB.



is $\frac{1}{4}$ OB

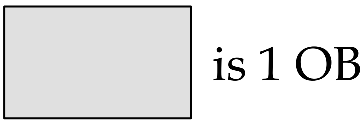
Unit of Re-measure



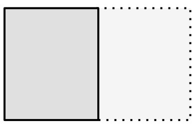
The re-measured length is 2 *of the* $\frac{1}{4}$ *OB*.

$$\frac{\frac{1}{2}\ OB}{\frac{1}{4}\ OB} = 2 \text{ and } 2 \times \frac{1}{4}\ OB = \frac{1}{2}\ OB$$

If the measured length is $\frac{1}{2}$ OB and it is re-measured using 4 OB, $\frac{\frac{1}{2}\ OB}{4\ OB}$, then the result is $\frac{1}{8}$ because the length $\frac{1}{2}$ OB is $\frac{1}{8}$ times as long as 4 OB ($\frac{1}{8}$ x 4OB = $\frac{1}{2}$ OB).

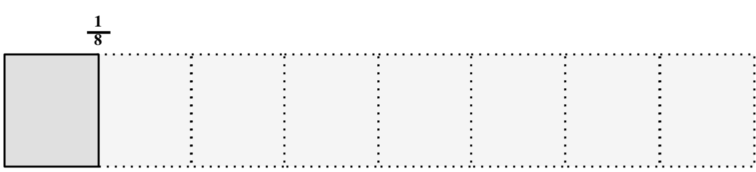


Original unit of measure



Original measured length is $\frac{1}{2}$ OB.

Unit of Re-measure is 4 OB.



Re-measured length is $\frac{1}{8}$ *of 4OB*.

$$\frac{\frac{1}{2}\ OB}{4\ OB} = \frac{1}{8} \quad \frac{\frac{1}{2}\ OB}{\frac{8}{2}\ OB} = \frac{1}{8} \quad \text{and} \quad \frac{1}{8} \times 4\ OB = \frac{4}{8}\ OB \text{ or } \frac{1}{2}\ OB$$

Creating An Algorithm. To create an algorithm, the relation k, between two measured lengths is the ratio of their common measure. For example:

$\frac{6\ in}{4\ in} = 1\frac{1}{2}$ (A 6-inch length is measured with a 4-inch unit)

$\frac{\frac{3}{4}\ in}{\frac{1}{2}\ in} = \frac{\frac{3}{4}\ in}{\frac{2}{4}\ in} = \frac{3}{2}$ (A $\frac{3}{4}$ in length is re-measured by a $\frac{1}{2}$ in unit)

$\frac{\frac{12}{10}\ miles}{\frac{4}{5}\ miles} = \frac{\frac{6}{5}\ miles}{\frac{4}{5}\ miles} = \frac{6}{4}$, and $\frac{6}{4} \times \frac{4}{5}\ miles = \frac{12}{10}\ miles$

Modeling Re-Measuring

Whole Group

1. **Hold a strip that is 1 foot long and place it on the whiteboard. Then iterate another 1 foot long strip 5 times, marking each iteration. Ask:**

   Q: How far did I travel, starting here and ending up here? What is the measure of the distance (5 feet). What was the unit of measure? (1 foot). 5 feet is _____ times as long as 1 *ft*.

2. **If I re-measure the same 5 *ft* distance with a unit that is less than 1 foot long, will the new measure be more, less or the same? Elicit predictions and justifications (less length covered by the unit, need more of them).**

3. **Fold a foot-strip to make $\frac{1}{2}$ *ft*. Place it on the whiteboard underneath the 1 *ft* strip. Let's think about re-measuring this distance with this unit that is $\frac{1}{2}$ times as long as 1 *ft*. Ask:**

   Q: What is the new measure? (10 of $\frac{1}{2}$ *ft*)

   Q: Has the distance changed? (No)

   Q: What has changed? (Its measure)

   We can represent what we did like this: $\frac{5\,ft}{\frac{1}{2}ft} = 10$.

4. **To restore the original measure, $10 \times \frac{1}{2} ft = 5\ ft$. It tells us that 5 *ft* is 10 times as long as $\frac{1}{2} ft$, or that there are 10 $\left(\frac{1}{2}\ ft\right)$'s in 5 *ft*.**

5. **If we re-measure the 5 *ft*, distance again, but this time with a unit that is more than 1 *ft* long, will the new measure be more, less or the same? Elicit predictions and justifications (more length covered by the unit, need fewer units to span the same distance).**

6. **Tape 5 foot-strips together. Place one on the whiteboard. Let's think about re-measuring this distance with a unit that is 2 times as long as 1 *ft*. We could call it a double-*ft*. Ask:**

   Q: What is the new measure? ($2\frac{1}{2}$ 2-*ft*)
   Q: Has the distance changed? (No)
   Q: What has changed? (Its measure)
   Q: 5 *ft* is ___ times as long as 2 ft.

   We can represent what we did like this: $\frac{5\,ft}{2\,ft} = 2\frac{1}{2}$.

7. **How can we get back to the original measure of 5 *ft*?**

   $2\frac{1}{2} \times 2\,ft = 2 \times 2\,ft + \frac{1}{2} \times 2\,ft = 5\,ft$ (Enact literally, if necessary)

   Teacher note: We are using the distributive property of multiplication over addition, as in (a + b) × c = ac + bc. Recall that $2\frac{1}{2}$ means $2 + \frac{1}{2}$.

Using Re-Measuring to Find Quotients for $\frac{a}{b}$ where b < a.

Partner / Individual

With a partner, or working by yourself, think of each of these as re-measuring the original length expressed by the numerator with the length expressed by the denominator. Find the result. What does it mean? Then check your answer by multiplying. (Each student should have a foot ruler) Make drawings to help you think.

$$\frac{5\ in}{4\ in}$$

$$\frac{6\ in}{\frac{1}{4}\ in}$$

$$\frac{\frac{1}{2}ft}{\frac{1}{4}ft}$$

$$\frac{\frac{5}{3}ft}{\frac{1}{3}ft}$$

$$\frac{6\ in}{3\ in}$$

$$\frac{20\ in}{4\ in}$$

Whole Class Discussion

Students share solution strategies. Be sure to enact and visualize each problem by re-measuring the numerator by the denominator. Emphasize that the resulting scalar multiple × the denominator-measure restores the original measure. This is the inverse relation between multiplication and division.

Using Re-Measuring to Find Quotients for $\frac{a}{b}$ where b > a.

Partner / Individual

With a partner, or work by yourself, think of each of these as re-measuring and find the result. Then check your answer. (Each student should have a one foot ruler). Make drawings to help you think.

$$\frac{\frac{1}{2}\,in}{4\,in}$$

$$\frac{\frac{1}{2}\,in}{2\,in}$$

$$\frac{\frac{1}{2}\,in}{1\,in}$$

$$\frac{\frac{1}{2}\,in}{\frac{3}{4}\,in}$$

Whole Class Discussion

Students share solution strategies. Be sure to enact and visualize each problem by re-measuring the numerator by the denominator. Emphasize that the resulting scalar multiple (a relationship number, also called a real; number) × the denominator measure restores the original measure (the inverse relation between multiplication and division).

Circle Measures

Partner Work

Using a compass, or a pencil and a string, construct a circle on paper. It can be as small or as large as you like. Be sure to mark its center. Use string to find the length of the circumference of the circle. Then cut a string to represent the length of a radius.

If the radius is the unit of measure, how many radius lengths are in one circumference length? Mark each radius length on the circumference of the circle.

Teacher note. There will be about 6.28 radius lengths in each circumference or about $6\frac{1}{4}$ radii. Another approach to this problem would be to have students measure each length in mm and then find k.

Cut a string to represent the length of the diameter of the circle. If the diameter length is used as the unit, what is the measure of the circumference? Mark each diameter length on the circumference of the circle.

Teacher note. There will be about 3.14 diameter lengths in each circumference or about $3\frac{1}{8}$ diameters. This measure has a special name, pi, symbolized by π.

Whole Group

Create a whole-class data table representing the findings of each partner pair for the radius and the diameter units of measure. Be sure to stress that not all the circles were exactly the same, but the relationship between circumference and radius or diameter is about the same. Elicit from students that the differences are likely due to measurement error.

Optional Extension

Draw an arc from the center of the circle that is 1 radius length long. The amount of turn represented by this arc is called a radian and is another measure of angle. Because there are 2π radii in the circumference of each circle, the angle measure of one whole rotation is 2π radians.

A Division Algorithm

Whole Group

1. **If we did not have strips and we wanted to know the results of re-measuring with a new unit, what method could we use that would work all the time?**

   Students may generalize from the examples involving common denominators and notice that the result of measuring with common measures is the ratio of the numerators. If no student suggests this generalization, you might wish to point to problems 5 and 7 of Modeling Division and ask what some of the other problems might look like if they were expressed in the same measurement unit (had the same denominators).

   A second level of support might be to re-express $\frac{3\ u}{\frac{1}{4}u}$ as $\frac{\frac{12}{4}u}{\frac{1}{4}u} = 12$

   (which can be verified by iterating $\frac{1}{4}$ u 12 times to establish that it is congruent with 3 u).

   The algorithm that we seek to promote is based on expressing the original and new units with a common unit. For example, $\frac{\frac{1}{2}}{2}$ can be re-expressed as $\frac{\frac{1}{2}}{\frac{4}{2}}$ with the resulting ratio of numerators $\frac{1}{4}$, because $\frac{1 \times \frac{1}{2}}{4 \times \frac{1}{2}} = \frac{1}{4}$.

   Similarly,

   $\frac{5}{\frac{1}{2}}$ can be re-expressed as $\frac{\frac{10}{2}}{\frac{1}{2}}$ for a resulting ratio of numerators of $\frac{10}{1}$.

Formative Assessment

NAME________________________________

1. A length has a measure of 6 inches. It is re-measured with a unit that is $\frac{1}{2}$ inch. What is its measure now? How could you check your answer?

2. $\frac{5\ in}{\frac{1}{8}\ in}$

Make a drawing that shows why your answer must be correct.

3. $\dfrac{\frac{4}{8}\,in}{\frac{2}{4}\,in}$

Make a drawing that shows why your answer must be correct.

4. $\dfrac{\frac{3}{4}\,in}{\frac{3}{2}\,in}$

Make a drawing that shows why your answer must be correct.

5. Use an algorithm to find $\frac{\frac{5}{12}}{\frac{1}{3}}$. Show your work.

6. What is the measure of the circumference of a circle if the length of the radius is used as the unit of measure? Explain how you know.

Student ____ Date ______________

For each student, indicate

Level	Description	Notes
Interprets $\frac{a}{b}$ as re-measuring (*a units*) in units of (*b units)* and can use algorithm.	All quotients are correct for items 1-4 and drawings show iteration of *b units* superimposed on *a units* for items 2-4. Shows understanding of inverse relation by using it to "check" results for item 1. Can find quotient in item 5 by using algorithm.	
Interprets $\frac{a}{b}$ as re-measuring (*a units*) in units of (*b units)*	All quotients are correct for items 1-4 and drawings show iteration of *b units* superimposed on *a units* for items 2-4. Shows understanding of inverse relation by using it to "check" results for item 1.	
Interprets $\frac{a}{b}$ as re-measuring (*a units*) in units of (*b units* when *b* < *a*	Items 1, 2, 3 quotients are correct and drawings show iteration of *b units* superimposed on *a units.*	
Partial understandings of $\frac{a}{b}$	Obtains quotients but drawings do not correspond to re-measurement interpretation.	
Emergent understanding	Describe.	
NL	Cannot interpret the meaning of fraction division	

Formative Assessment Record

Circle Measures Indicate student responses and quality of explanation.